Feynman Rules for Any Spin. II. Massless Particles*

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(Received 13 January 1964)

The Feynman rules are derived for massless particles of arbitrary spin j. The rules are the same as those presented in an earlier article for $m>0$, provided that we let $m \to 0$ in propagators and wave functions, and provided that we keep to the $(2j+1)$-component formalism [with fields of the $(j,0)$ or $(0,j)$ type] or the $2(2j+1)$-component formalism [with $(j,0)\oplus(0,j)$ fields]. But there are other field types which cannot be constructed for $m=0$; these include the $(j/2,j/2)$ tensor fields, and in particular the vector potential for $j=1$. This restriction arises from the non-semi-simple structure of the little group for $m=0$. Some other subjects discussed include: **T**, **C**, and **P** for massless particles and fields; the extent to which chirality conservation implies zero physical mass; and the Feynman rules for massive particles in the helicity formalism. Our approach is based on the assumption that the S matrix is Lorentz invariant, and makes no use of Lagrangians or the canonical formalism.

I. INTRODUCTION

THIS article will develop the relativistic field theory of massless particles with general spin, along the lines followed in an earlier work[1] on massive particles. Our chief aim is, again, to derive the Feynman rules.

We assume that the S matrix can be calculated from Dyson's formula

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\cdots d^4x_n T\{\mathcal{H}(x_1)\cdots\mathcal{H}(x_n)\}\,. \quad (1.1)$$

Here, $\mathcal{H}(x)$ is the interaction energy density in the interaction representation. In general, it would be the 00 component $\mathcal{T}^{00}(x)$ of a tensor $\mathcal{T}^{\mu\nu}(x)$, but in order that S be Lorentz-invariant it is necessary that $\mathcal{T}^{\mu\nu}(x)$ be of the form

$$\mathcal{T}^{\mu\nu}(x)=-g^{\mu\nu}\mathcal{H}(x)\,, \quad (1.2)$$

with $\mathcal{H}(x)$ a scalar. Lorentz invariance also dictates that $\mathcal{H}(x)$ commute with $\mathcal{H}(y)$ for $x-y$ space-like, in order that the θ functions implicit in the time-ordered product in (1.1) not destroy the Lorentz invariance of S.

We also assume that $\mathcal{H}(x)$ is built out of the creation and annihilation operators of the free particles appearing in the unperturbed Hamiltonian. In order that $\mathcal{H}(x)$ transform properly we construct it as an invariant polynomial in various free fields $\psi_n(x)$, which behave as usual under translations, and which transform according to various representations of the homogeneous Lorentz group

$$U[\Lambda]\psi_n(x)U[\Lambda]^{-1}=\sum_m D_{nm}[\Lambda^{-1}]\psi_m(\Lambda x)\,. \quad (1.3)$$

In order that $\mathcal{H}(x)$ commute with itself outside the light cone, we require that the $\psi_n(x)$ have causal commutation or anticommutation rules: for $x-y$ space-like,

$$[\psi_n(x),\psi_m(y)]_\pm=0\,. \quad (1.4)$$

These assumptions will be sufficient for all our purposes. In particular, we will have no need of Lagrangians and the canonical formalism, nor will we need to start with any preconceptions about the form or even the existence of the field equations.

We begin in Sec. II with a review of the transformation properties of massless particle states and creation and annihilation operators. This information is used in Secs. III and IV to construct $(2j+1)$-component fields transforming according to the $(j,0)$ and $(0,j)$ representations. Condition (1.4) is used in Sec. V to complete the construction of the fields, and to prove the spin-statistics theorem and crossing symmetry. The Feynman rules are presented in Secs. VI, VII, and VIII. The inversions **P**, **C**, and **T** are discussed in Sec. IX.

In Sec. X we attack a separate problem: To what extent does chirality conservation guarantee the existence of a particle of zero physical mass? Our conclusion [for general $j\geq\frac{1}{2}$] is that this theorem can probably only be proved in the context of perturbation theory. But if parity as well as chirality is conserved, then it is possible to prove the nonexistence of a nondegenerate particle of finite mass.

The chief conclusion of this work is that the Feynman rules for massless particles in the $(2j+1)$-component or $2(2j+1)$-component formalisms are precisely the same as for $m>0$, except, of course, that we must pass to the limit $m\to 0$ in wave functions and propagators.[2] In this limit it becomes impossible to produce or destroy particles with helicity other than $\pm j$.

But there is still one important qualitative distinction between $m=0$ and $m>0$. We prove in Sec. III that not all of the field types which can be constructed out of the creation and annihilation operators for $m>0$ can be so constructed for $m=0$. Specifically, the annihilation operator for a massless particle of helicity λ and the

* Research supported in part by the U. S. Air Force Office of Scientific Research, Grant No. AF-AFOSR-232-63.
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1 S. Weinberg, Phys. Rev. **133**, B1318 (1964).

2 This conclusion is in agreement with the theorem that the decomposition of the S matrix into invariant amplitudes takes the same form for $m=0$ and $m>0$, proven by D. Zwanziger, Phys. Rev. **133**, B1036 (1964). Neither Zwanziger's work nor the present article offer any understanding of the fact that photons and gravitons interact with conserved quantities at zero-momentum transfer. This point will be the subject of further articles, to be published in Phys. Letters and in Phys. Rev.

creation operator for the antiparticle with helicity $-\lambda$ can only be used to form a field transforming as in (1.3) under those representations (A,B) of the homogeneous Lorentz group such that $\lambda=B-A$. This limitation arises purely because of the non-semi-simple structure of the little group for $m=0$. The difficulties (indefinite metric, negative energies, etc.) encountered in previous attempts to represent the photon by a quantized vector potential $A^\mu(x)$ can therefore now be understood as due to the fact that such a field transforms according to the $(\frac{1}{2},\frac{1}{2})$ representation, which is not one of the representations allowed by the theorem of Sec. III for helicity $\lambda=\pm1$. On the other hand, the $(j,0)$ and $(0,j)$ representations used in this article (corresponding for $j=1$ to the field strengths) are allowed by our theorem, and they cause no trouble.[3] In a future article we shall show that it is in fact possible to evade our theorem, and that the Lorentz invariance of the S matrix then forces us to the principle of extended gauge invariance.

In Ref. 1 we gave the Feynman rules for initial and final states specified by the z components of the massive particle spins. In order to facilitate the comparison with the case of zero mass, and for the sake of completeness, we present in Sec. VIII the corresponding Feynman rules in the helicity formalism of Jacob and Wick.[4] The external-line wave functions are much simpler, though of course the propagators are the same.

II. TRANSFORMATION OF STATES

The starting point in our approach is a statement of the Lorentz transformation properties of massless particle *states*. The transformation rules have been completely worked out by Wigner,[5] but it will be convenient to review them here, particularly as there are some little known but extremely important peculiarities that are special to the case of zero mass.

Consider a massless particle moving in the z direction with energy κ. It may have several possible spin states, which we denote $|\lambda\rangle$, the significance of the label λ to be determined by examining the transformation properties of these states. Wigner defines the "little group" as the subgroup of the Lorentz group consisting of all homogeneous proper Lorentz transformations $\mathcal{R}^\mu{}_\nu$ which do not alter the four-momentum k^μ of our particle.

$$\mathcal{R}^\mu{}_\nu k^\nu=k^\mu\,, \tag{2.1}$$

$$k^1=k^2=0\,;\quad k^3=k^0=\kappa\,. \tag{2.2}$$

[3] As a case in point, there does not seem to be any obstacle to the construction of field theories for massless charged particles of arbitrary spin j, provided that we use only proper field types, like $(j,0)$ or $(0,j)$. The trouble encountered for $j\geqq1$ by K. M. Case and S. G. Gasiorowicz [Phys. Rev. **125**, 1055 (1962)], can be ascribed to their use of improper field types, such as $(\frac{1}{2},\frac{1}{2})$. We plan to discuss this in more detail in a later article on the electromagnetic interactions of particles of any spin.

[4] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[5] E. P. Wigner, in *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963), p. 59.

The states $|\lambda\rangle$ must furnish a representation of the little group. That is, the unitary operator $U[\mathcal{R}]$ corresponding to $\mathcal{R}^\mu{}_\nu$ does not change the momentum of the states $|\lambda\rangle$, and thus must just induce a linear transformation:

$$U[\mathcal{R}]|\lambda\rangle=\sum_{\lambda'} d_{\lambda'\lambda}[\mathcal{R}]|\lambda'\rangle\,, \tag{2.3}$$

with

$$\sum_{\lambda''} d_{\lambda\lambda''}[\mathcal{R}_1]d_{\lambda''\lambda'}[\mathcal{R}_2]=d_{\lambda\lambda'}[\mathcal{R}_1\mathcal{R}_2]\,. \tag{2.4}$$

Therefore, we can catalog the various possible spin states $|\lambda\rangle$ by studying the representations $d[\mathcal{R}]$ of the little group.

This is most easily accomplished by examining the infinitesimal transformations of the little group. They take the form

$$\mathcal{R}^\mu{}_\nu=\delta^\mu{}_\nu+\Omega^\mu{}_\nu\,, \tag{2.5}$$

where $\Omega^\mu{}_\nu$ is infinitesimal and annihilates k:

$$\Omega^\mu{}_\nu k^\nu=0\,. \tag{2.6}$$

In order that (2.5) be a Lorentz transformation we must also require that

$$\Omega^{\mu\nu}=-\Omega^{\nu\mu}\,, \tag{2.7}$$

the index ν being raised in the usual way with the metric tensor $g^{\mu\nu}$, defined here to have nonzero components:

$$g^{11}=g^{22}=g^{33}=1\,,\quad g^{00}=-1\,. \tag{2.8}$$

Inspection of (2.6) and (2.7) shows that the general $\Omega^{\mu\nu}$ is a function of three parameters θ, χ_1, χ_2, with nonzero components given by

$$\Omega^{12}=-\Omega^{21}=\theta\,, \tag{2.9}$$

$$\Omega^{10}=-\Omega^{01}=\Omega^{13}=-\Omega^{31}=\chi_1\,, \tag{2.10}$$

$$\Omega^{20}=-\Omega^{02}=\Omega^{23}=-\Omega^{32}=\chi_2\,. \tag{2.11}$$

The Lie algebra generated by these transformations can be determined by recalling the algebra generated by the full homogeneous Lorentz group, of which the little group is a subgroup. An infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu$ can be written as in (2.5), with $\Omega^\mu{}_\nu$ subject only to (2.7). The corresponding unitary operator takes the form

$$U[1+\Omega]=1+(i/2)\Omega^{\mu\nu}J_{\mu\nu}\,, \tag{2.12}$$

$$J_{\mu\nu}=-J_{\nu\mu}=J_{\mu\nu}{}^\dagger\,. \tag{2.13}$$

It is conventional to group the six components of $J_{\mu\nu}$ into two three-vectors:

$$J_i=\tfrac{1}{2}\epsilon_{ijk}J_{jk}\,, \tag{2.14}$$

$$K_i=J_{i0}=-J_{0i}\,, \tag{2.15}$$

with commutation rules

$$[J_i,J_j]=i\epsilon_{ijk}J_k\,, \tag{2.16}$$

$$[J_i,K_j]=i\epsilon_{ijk}K_k\,, \tag{2.17}$$

$$[K_i,K_j]=-i\epsilon_{ijk}J_k\,. \tag{2.18}$$

We see that the unitary operator corresponding to the general infinitesimal transformation (2.9)–(2.11) of the little group is

$$U[\mathcal{R}(\theta,\chi_1,\chi_2)]=1+i\theta J_3+i\chi_1L_1+i\chi_2L_2\,, \tag{2.19}$$

where

$$L_1\equiv K_1-J_2\,, \tag{2.20}$$

$$L_2\equiv K_2+J_1\,. \tag{2.21}$$

The commutation rules for the three generators of the little group are given by (2.16)–(2.18) as

$$[J_3,L_1]=iL_2\,, \tag{2.22}$$

$$[J_3,L_2]=-iL_1\,, \tag{2.23}$$

$$[L_1,L_2]=0\,. \tag{2.24}$$

We can now find all the representations of the little group by finding the representations of this Lie algebra. But it strikes one immediately that this algebra is not semi-simple because the elements L_1 and L_2 form an invariant Abelian subalgebra. [In fact, Wigner[5] points out that (2.22)–(2.24) identify this algebra as that of all rotations *and translations* in two-dimensions, a fact of no known physical significance.] In order that the states $|\lambda\rangle$ form a finite set, it is necessary to represent the "translations" by zero, i.e.,

$$L_1|\lambda\rangle=L_2|\lambda\rangle=0\ . \tag{2.25}$$

Therefore, a general $\mathcal{R}^\mu{}_\nu$ in the little group transforms $|\lambda\rangle$ into

$$U[\mathcal{R}]|\lambda\rangle=\exp\{i\Theta[\mathcal{R}]J_3\}|\lambda\rangle\,, \tag{2.26}$$

the angle $\Theta[\mathcal{R}]$ being some more or less complicated real function of the $\mathcal{R}^\mu{}_\nu$, which is given for infinitesimal $\mathcal{R}$ by (2.19) as

$$\Theta[\mathcal{R}(\theta,\chi_1,\chi_2)]\to\theta\,. \tag{2.27}$$

If we now identify the states $|\lambda\rangle$ as eigenstates with definite helicity λ,

$$J_3|\lambda\rangle=\lambda|\lambda\rangle\,, \tag{2.28}$$

we see that the physically permissible irreducible representations of the little group are all one dimensional:

$$U[\mathcal{R}]|\lambda\rangle=\exp\{i\lambda\Theta[\mathcal{R}]\}|\lambda\rangle\,. \tag{2.29}$$

Comparing with (2.3) and (2.4) shows that Θ must satisfy the group property

$$\Theta[\mathcal{R}_1]+\Theta[\mathcal{R}_2]=\Theta[\mathcal{R}_1\mathcal{R}_2]\,. \tag{2.30}$$

For global reasons it is necessary to restrict the helicity λ to be a positive or negative integer or half-integer $\pm j$. We *define* a right- or left-handed particle of spin $j\geqq 0$ as one with helicity λ equal to $+j$ or $-j$, respectively.

It is, of course, very well known that a spinning massless particle need not occur in more than one spin state (or two, if parity is conserved). The restriction (2.25) is much less familiar, but we shall see that it is responsible for the *dynamical* peculiarities of massless particle field theories.

A particle of general momentum $\mathbf{p}$ and helicity λ may now be defined by a Lorentz transformation

$$|\mathbf{p},\lambda\rangle\equiv[\kappa/|\mathbf{p}|]^{1/2}U[\mathcal{L}(\mathbf{p})]|\lambda\rangle\,, \tag{2.31}$$

where $U[\mathcal{L}(\mathbf{p})]$ is the unitary operator corresponding to the Lorentz transformation $\mathcal{L}_\nu{}^\mu(\mathbf{p})$ which takes our "standard" four-momentum k^μ into p^μ:

$$\begin{gathered} p^\mu=\mathcal{L}^\mu{}_\nu(\mathbf{p})k^\nu\,,\\ p^\mu=\{\mathbf{p},|\mathbf{p}|\}\,;\quad k^\mu=\{0,0,\kappa,\kappa,\}\,. \end{gathered} \tag{2.32}$$

There are various ways of making the definition of $\mathcal{L}(\mathbf{p})$ unambiguous, but we will find it convenient to define $\mathcal{L}$ as

$$\mathcal{L}^\mu{}_\nu(\mathbf{p})=R^\mu{}_\lambda(\hat{p})B^\lambda{}_\nu(|\mathbf{p}|)\,. \tag{2.33}$$

Here, $B(|\mathbf{p}|)$ is a "boost" along the z axis with nonzero components

$$\begin{aligned} &B^1{}_1(|\mathbf{p}|)=B^2{}_2(|\mathbf{p}|)=1\,,\\ &B^3{}_3(|\mathbf{p}|)=B^0{}_0(|\mathbf{p}|)=\cosh\phi(|\mathbf{p}|)\,, \end{aligned} \tag{2.34}$$

$$\begin{aligned} &B^3{}_0(|\mathbf{p}|)=B^0{}_3(|\mathbf{p}|)=\sinh\phi(|\mathbf{p}|)\,,\\ &\phi(|\mathbf{p}|)\equiv\ln(|\mathbf{p}|/\kappa)\,. \end{aligned} \tag{2.35}$$

Since $B^\mu{}_\nu$ takes k^μ into $\{0,0,|\mathbf{p}|,|\mathbf{p}|\}$, we choose $R(\hat{p})$ as the rotation (say, in the plane containing $\hat{p}$ and the z axis) which takes the z axis into the unit vector $\hat{p}=\mathbf{p}/|\mathbf{p}|$. The factor $[\kappa/|\mathbf{p}|]^{1/2}$ is inserted in (2.31) to keep the normalization conventional,

$$\langle\mathbf{p}',\lambda'|\mathbf{p},\lambda\rangle=\delta^3(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}\,. \tag{2.36}$$

Having defined helicity states of arbitrary momentum in terms of states $|\lambda\rangle$ of a fixed standard four-momentum k^μ, it is now quite easy to find their transformation properties. A general Lorentz transformation $\Lambda^\mu{}_\nu$, represented on Hilbert space by a unitary operator $U[\Lambda]$, will transform $|\mathbf{p},\lambda\rangle$ into

$$\begin{aligned} U[\Lambda]|\mathbf{p},\lambda\rangle &=[\kappa/|\mathbf{p}|]^{1/2}U[\Lambda]U[\mathcal{L}(\mathbf{p})]|\lambda\rangle\\ &=[\kappa/|\mathbf{p}|]^{1/2}U[\mathcal{L}(\Lambda\mathbf{p})]U[\mathcal{L}^{-1}(\Lambda\mathbf{p})\Lambda\mathcal{L}(\mathbf{p})]|\lambda\rangle\,. \end{aligned} \tag{2.37}$$

But the transformation $\mathcal{L}^{-1}(\Lambda\mathbf{p})\Lambda\mathcal{L}(\mathbf{p})$ leaves k^μ unchanged, and hence belongs to the little group. Equation (2.29) then lets us write (2.37) as

$$U[\Lambda]|\mathbf{p},\lambda\rangle=[\kappa/|\mathbf{p}|]^{1/2}\times\exp\{i\lambda\Theta[\mathcal{L}^{-1}(\Lambda\mathbf{p})\Lambda\mathcal{L}(\mathbf{p})]\}U[\mathcal{L}(\Lambda\mathbf{p})]|\lambda\rangle\,,$$

and finally

$$U[\Lambda]|\mathbf{p},\lambda\rangle=[|\Lambda\mathbf{p}|/|\mathbf{p}|]^{1/2}\times\exp\{i\lambda\Theta[\mathcal{L}^{-1}(\Lambda\mathbf{p})\Lambda\mathcal{L}(\mathbf{p})]\}|\Lambda\mathbf{p},\lambda\rangle\,. \tag{2.38}$$

A general state containing several free particles will transform like (2.38), with a factor $[|\mathbf{p}'|/|\mathbf{p}|]^{1/2}e^{i\Theta\lambda}$ for each particle. These states can be built up by acting on the bare vacuum with creation operators $a^*(\mathbf{p},\lambda)$ which satisfy either the usual Bose or Fermi rules:

$$[a(\mathbf{p},\lambda),a^*(\mathbf{p}',\lambda')]_\pm=\delta_{\lambda\lambda'}\delta^3(\mathbf{p}-\mathbf{p}')\,, \qquad (2.39)$$

so the general transformation law can be summarized in the statement

$$U[\Lambda]a^*(\mathbf{p},\lambda)U^{-1}[\Lambda]=[|\Lambda\mathbf{p}|/|\mathbf{p}|]^{1/2} \times\exp\{i\lambda\Theta[\mathcal{L}^{-1}(\Lambda\mathbf{p})\Lambda\mathcal{L}(\mathbf{p})]\}a^*(\Lambda\mathbf{p},\lambda)\,. \qquad (2.40)$$

Taking the adjoint and using the property [see (2.30)]

$$\Theta[R]=-\Theta[R^{-1}] \qquad (2.41)$$

gives the transformation rule of the annihilation operator

$$U[\Lambda]a(\mathbf{p},\lambda)U^{-1}[\Lambda]=[|\Lambda\mathbf{p}|/|\mathbf{p}|]^{1/2} \times\exp\{i\lambda\Theta[\mathcal{L}^{-1}(\mathbf{p})\Lambda^{-1}\mathcal{L}(\Lambda\mathbf{p})]\}a(\Lambda\mathbf{p},\lambda)\,. \qquad (2.42)$$

We speak of one massless particle as being the antiparticle of another if their spins j are the same, while all their charges, baryon numbers, etc., are equal and opposite. Whether or not every massless particle has such an antiparticle is an open question, to be answered affirmatively in Sec. V. But if an antiparticle exists, then its creation operator $b^*(\mathbf{p},\lambda)$ will transform just like $a^*(\mathbf{p},\lambda)$, and $b^*(\mathbf{p},-\lambda)$ will transform just like $a(\mathbf{p},\lambda)$:

$$U[\Lambda]b^*(\mathbf{p},-\lambda)U^{-1}[\Lambda]=[|\Lambda\mathbf{p}|/|\mathbf{p}|]^{1/2} \times\exp\{i\lambda\Theta[\mathcal{L}^{-1}(\mathbf{p})\Lambda^{-1}\mathcal{L}(\Lambda\mathbf{p})]\}b^*(\Lambda\mathbf{p},-\lambda)\,. \qquad (2.43)$$

If a particle is its own antiparticle,[6] then we just set $b(\mathbf{p},\lambda)=a(\mathbf{p},\lambda)$.

III. A THEOREM ON GENERAL FIELDS

As a first step, let us try to construct the "annihilation fields" $\psi_n^{(+)}(x;\lambda)$, as linear combinations of the annihilation operators $a(p,\lambda)$, with fixed helicity λ. We require that the $\psi_n^{(+)}$ transform as usual under translations

$$i[P_\mu,\psi_n^{(+)}(x;\lambda)]=\partial_\mu\psi_n^{(+)}(x;\lambda) \qquad (3.1)$$

and transform according to some irreducible representation $D[\Lambda]$ of the *homogeneous* proper orthochronous Lorentz group:

$$U[\Lambda]\psi_n^{(+)}(x;\lambda)U[\Lambda]^{-1} =\sum_m D_{nm}[\Lambda^{-1}]\psi_m^{(+)}(\Lambda x;\lambda)\,. \qquad (3.2)$$

It is well known that the various representations $D[\Lambda]$ can be cataloged by writing the matrices $\boldsymbol{J}$ and $\boldsymbol{K}$, which represent the rotation generator $\mathbf{J}$ and the boost generator $\mathbf{K}$ as

$$\boldsymbol{J}=\boldsymbol{A}+\boldsymbol{B}\,;\quad \boldsymbol{K}=-i(\boldsymbol{A}-\boldsymbol{B})\,. \qquad (3.3)$$

Since $\boldsymbol{J}$ and $\boldsymbol{K}$ satisfy the same commutation rules (2.16)–(2.18) as $\mathbf{J}$ and $\mathbf{K}$, the $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy decoupled commutation rules

$$\boldsymbol{A}\times\boldsymbol{A}=i\boldsymbol{A}\,;\quad \boldsymbol{B}\times\boldsymbol{B}=i\boldsymbol{B}\,, \qquad [\mathcal{A}_i,\mathcal{B}_j]=0\,. \qquad (3.4)$$

The general $(2A+1)(2B+1)$-dimensional irreducible representation (A,B) is conventionally defined for integer values of $2A$ and $2B$ by

$$\boldsymbol{A}_{ab,a'b'}=\delta_{bb'}\mathbf{J}_{aa'}^{(A)}\,, \qquad \boldsymbol{B}_{ab,a'b'}=\delta_{aa'}\mathbf{J}_{bb'}^{(B)}\,, \qquad (3.5)$$

where a and b run by unit steps from $-A$ to $+A$ and from $-B$ to $+B$, respectively, and $J^{(j)}$ is the usual $2j+1$-dimensional representation of the angular momentum

$$[J_1^{(j)}\pm iJ_2^{(j)}]_{\sigma'\sigma}=\delta_{\sigma',\sigma\pm1}[(j\mp\sigma)(j\pm\sigma+1)]^{1/2}\,, \qquad [J_3^{(j)}]_{\sigma'\sigma}=\sigma\delta_{\sigma'\sigma}\,. \qquad (3.6)$$

For massive particles of spin j, we have already seen in Sec. VIII of Ref. 1 that a field $\psi^{(+)}(x)$ can be constructed out of the $2j+1$ annihilation operators $a(\mathbf{p},\sigma)$, which will satisfy the transformation requirements (3.1) and (3.2), for any representation (A,B) that "contains" j, i.e., such that

$$j=A+B \text{ or } A+B-1 \text{ or } \cdots \text{ or } |A-B|\,. \qquad (3.7)$$

[A spin-one field could be a four-vector $(\frac{1}{2},\frac{1}{2})$, a tensor (1,0) or (0,1), etc.] We might expect the same to be true for mass zero, *but this is not the case.* We will prove in this section that a massless particle operator $a(\mathbf{p},\lambda)$ of helicity λ can only be used to construct fields which transform according to representations (A,B) such that

$$B-A=\lambda\,. \qquad (3.8)$$

For instance, a left-circularly polarized photon with $\lambda=-1$ can be associated with (1,0), $(\frac{3}{2},\frac{1}{2})$, (2,1), $\cdots$ fields but *not* with the vector potential $(\frac{1}{2},\frac{1}{2})$, at least until we broaden our notion of what we mean by a Lorentz transformation. It will be seen that the restriction (3.8) arises because of the non-semi-simple structure of the little group.

[6] It is not so obvious what is meant by a massless particle being its own antiparticle. If charge conjugation were conserved, then we would call a particle purely neutral if it were invariant (up to a phase) under **C**. But if we take weak interactions into account then only **CP** and **CPT** are available, and they convert a particle into the antiparticle with opposite helicity. For massless particles there is no way of deciding whether a particle is the "same" as another of opposite helicity, since one cannot be converted into the other by a rotation. This point has been thoroughly explored with regard to the neutrino by J. A. McLennan, Phys. Rev. **106**, 821 (1957) and K. M. Case, *ibid.* **107**, 307 (1957). See also C. Ryan and S. Okubo, Rochester Preprint URPA-3 (to be published). Even if a massless particle carries some quantum number (like lepton number), we can still call it purely neutral if we let its quantum number depend on the helicity; however, in this case it seems more natural to adopt the convention that the particle is different from its antiparticle, with $b(\mathbf{p},\lambda)\neq a(\mathbf{p},\lambda)$.

The condition (3.1) requires that $\psi_n^{(+)}$ be constructed as a Fourier transform

$$\psi_n^{(+)}(x;\lambda)=\frac{1}{(2\pi)^{3/2}}\times\int\frac{d^3p}{[2|p|]^{1/2}}e^{ip\cdot x}a(\mathbf{p},\lambda)u_n(\mathbf{p},\lambda)\,, \quad (3.9)$$

the factor $(2\pi)^{-3/2}[2|\mathbf{p}|]^{-1/2}$ being extracted from the "wave function" $u_n(\mathbf{p},\lambda)$ for later convenience. The condition (3.2) together with the transformation rule (2.42) then requires that $u_n(\mathbf{p},\lambda)$ satisfy

$$\exp\{i\lambda\Theta[\mathcal{L}^{-1}(\mathbf{p})\Lambda^{-1}\mathcal{L}(\Lambda\mathbf{p})]\}u_n(\mathbf{p},\lambda)=\sum_m D_{nm}[\Lambda^{-1}]u_m(\Lambda\mathbf{p},\lambda)\,. \quad (3.10)$$

We will now show that this determines $u_m(\mathbf{p},\lambda)$ uniquely.

In particular (3.10) must be satisfied if we choose

$$\mathbf{p}=\mathbf{k}\equiv\{0,0,\kappa\}\,;\ \Lambda=\mathcal{L}(\mathbf{q})\,,$$

where $\mathbf{q}$ is some arbitrary momentum. In this case (3.10) reads

$$u_n(\mathbf{q},\lambda)=\sum_m D_{nm}[\mathcal{L}(\mathbf{q})]u_m(\lambda)\,, \quad (3.11)$$

where $u_m(\lambda)$ is the wave function for our "standard" momentum $\mathbf{k}$

$$u_m(\lambda)\equiv u_m(\mathbf{k},\lambda)\,. \quad (3.12)$$

Insertion of (3.11) into both sides of (3.10) shows that (3.10) is satisfied by (3.11) if and only if the $u_m(\lambda)$ satisfy

$$\exp\{i\lambda\Theta[\mathcal{L}^{-1}(\mathbf{p})\Lambda^{-1}\mathcal{L}(\Lambda\mathbf{p})]\}\sum_m D_{nm}[\mathcal{L}(\mathbf{p})]u_m(\lambda)=\sum_m D_{nm}[\Lambda^{-1}\mathcal{L}(\Lambda\mathbf{p})]u_m(\lambda)\,,$$

or in other words, if and only if

$$\sum_m D_{nm}[\mathcal{R}]u_m(\lambda)=\exp\{i\lambda\Theta[\mathcal{R}]\}u_n(\lambda) \quad (3.13)$$

for any Lorentz transformation $\mathcal{R}$ of the form

$$\mathcal{R}=\mathcal{L}^{-1}(\mathbf{p})\Lambda^{-1}\mathcal{L}(\Lambda\mathbf{p})\,. \quad (3.14)$$

But these $\mathcal{R}$'s, for general $\mathbf{p}$ and Λ, just constitute the little group discussed in Sec. II. In order that (3.13) be satisfied for all such $\mathcal{R}$ it is necessary and sufficient that it be satisfied for all infinitesimal transformations

$$\mathcal{R}^\mu{}_\nu=\delta^\mu{}_\nu+\Omega^\mu{}_\nu(\theta,\chi_1,\chi_2)\,, \quad (3.15)$$

the nonvanishing components of Ω being given by (2.9)–(2.11). The matrix $D[\mathcal{R}]$ corresponding to (3.15) is obtained by replacing $\mathbf{J}$ and $\mathbf{K}$ in (2.19) by their matrix representatives J and K:

$$D[\mathcal{R}(\theta,\chi_1,\chi_2)]=1+i\theta\mathcal{J}_3+i\chi_1(\mathcal{K}_1-\mathcal{J}_2)+i\chi_2(\mathcal{K}_2+\mathcal{J}_1)\,, \quad (3.16)$$

or, using (3.3),

$$D[\mathcal{R}(\theta,\chi_1,\chi_2)]=1+i\theta(\mathcal{A}_3+\mathcal{B}_3)+(\chi_1+i\chi_2)(\mathcal{A}_1-i\mathcal{A}_2)+(\chi_1-i\chi_2)(\mathcal{B}_1+i\mathcal{B}_2)\,. \quad (3.17)$$

Recalling from (2.27) that $\Theta\to\theta$, our condition (3.13) is now split into three independent conditions:

$$[\mathcal{A}_3+\mathcal{B}_3]u(\lambda)=\lambda u(\lambda)\,, \quad (3.18)$$

$$[\mathcal{A}_1-i\mathcal{A}_2]u(\lambda)=0\,, \quad (3.19)$$

$$[\mathcal{B}_1+i\mathcal{B}_2]u(\lambda)=0\,. \quad (3.20)$$

Of these three conditions, (3.18) could certainly have been anticipated as necessary to a field of helicity λ. The other two arise from the detailed structure of the little group, but are equally important, for they force $u(\lambda)$ to be an eigenvector of $\mathcal{A}_3$ and $\mathcal{B}_3$, with

$$\mathcal{A}_3u(\lambda)=-Au(\lambda)\,, \quad (3.21)$$

$$\mathcal{B}_3u(\lambda)=+Bu(\lambda)\,, \quad (3.22)$$

or more explicitly

$$u_{ab}(\lambda)=\delta_{a,-A}\delta_{b,B}\,. \quad (3.23)$$

Using (3.18) now gives the promised restriction on A and B:

$$-A+B=\lambda\,. \quad (3.8)$$

For a left-handed particle with $\lambda=-j$, the various possible fields are

$$[\text{left}]\quad (j,0),\ (j+\tfrac{1}{2},\tfrac{1}{2}),\ (j+1,1),\ \cdots, \quad (3.24)$$

while a right-handed particle with $\lambda=+j$ can be associated with a field transforming like

$$[\text{right}]\quad (0,j),\ (\tfrac{1}{2},j+\tfrac{1}{2}),\ (1,j+1),\ \cdots. \quad (3.25)$$

If parity is conserved, then the particle must exist in both states $\lambda=\pm j$, and the field *must* then transform reducibly, for example, like $(j,0)\oplus(0,j)$.

Our theorem certainly applies to the *in* and *out* fields, since they are constructed just like free fields. It must then also apply to the Heisenberg representation field that interpolates between *in* and *out* fields if we insist that they all behave in the same way under Lorentz transformations. Furthermore, the only "*M* functions"[7] that can generally be formed from the S matrix are those corresponding to the representations (3.24) and (3.25).

In a forthcoming article we shall see what goes wrong when we try to construct a field with A and B violating (3.8).

[7] H. Stapp, Phys. Rev. **125**, 2139 (1962); A. O. Barut, I. Muzinich, and D. N. Williams, *ibid.* **130**, 442 (1963).

IV. $(2j+1)$-COMPONENT FIELDS

For a left- or right-handed particle with $\lambda=-j$ or $\lambda=+j$, the simplest field type listed in (3.24) or (3.25) is, respectively, $(j,0)$ or $(0,j)$. The corresponding $(2j+1)$-component annihilation fields will be called $\varphi_\sigma^{(+)}(x)$ and $\chi_\sigma^{(+)}(x)$. They are given by (3.9), (3.11), and (3.23), as

$$\varphi_\sigma^{(+)}(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{[2|\mathbf{p}|]^{1/2}}\times e^{ip\cdot x}D_{\sigma,-j}^{(j)}[\mathcal{L}(\mathbf{p})]a(\mathbf{p},-j)\,,\quad(4.1)$$

$$\chi_\sigma^{(+)}(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{[2|\mathbf{p}|]^{1/2}}\times e^{ip\cdot x}\bar{D}_{\sigma j}^{(j)}[\mathcal{L}(\mathbf{p})]a(\mathbf{p},j)\,,\quad(4.2)$$

and they transform according to

$$U[\Lambda]\varphi_\sigma^{(+)}(x)U^{-1}[\Lambda]=\sum_{\sigma'}D_{\sigma\sigma'}^{(j)}[\Lambda^{-1}]\varphi_{\sigma'}^{(+)}(\Lambda x)\,,\quad(4.3)$$

$$U[\Lambda]\chi_\sigma^{(+)}(x)U^{-1}[\Lambda]=\sum_{\sigma'}\bar{D}_{\sigma\sigma'}^{(j)}[\Lambda^{-1}]\chi_{\sigma'}^{(+)}(\Lambda x)\,.\quad(4.4)$$

Here $D^{(j)}[\Lambda]$ and $\bar{D}^{(j)}[\Lambda]$ are the nonunitary $(2j+1)\times(2j+1)$-dimensional matrices corresponding to Λ in the $(j,0)$ and $(0,j)$ representations, respectively. They are the same as used in Ref. 1, and can be defined by taking $\mathcal{B}=0$ or $\mathcal{A}=0$, or, equivalently, by representing the generators $\mathbf{J}$, $\mathbf{K}$ with

$$D^{(j)}:\quad \mathbf{J}=\mathbf{J}^{(j)}\,,\quad \mathbf{K}=-i\mathbf{J}^{(j)}\,,\quad(4.5)$$

$$\bar{D}^{(j)}:\quad \mathbf{J}=\mathbf{J}^{(j)}\,,\quad \mathbf{K}=+i\mathbf{J}^{(j)}\,,\quad(4.6)$$

where $\mathbf{J}^{(j)}$ is the usual spin-j representation of the angular momentum, defined by (3.6). In particular, the transformation $\mathcal{L}(\mathbf{p})$ defined by (2.33) is represented on Hilbert space by

$$U[\mathcal{L}(\mathbf{p})]=U[R(\hat{p})]\exp\{-i\phi(|\mathbf{p}|)K_3\}\,,\quad(4.7)$$

$$\phi(|\mathbf{p}|)=\ln[|\mathbf{p}|/\kappa]\,,\quad(2.35)$$

and therefore the wave functions appearing in (4.1) and (4.2) are

$$D_{\sigma,-j}^{(j)}[\mathcal{L}(\mathbf{p})]=\sum_{\sigma'}D_{\sigma\sigma'}^{(j)}[R(\hat{p})][\exp\{-\phi(|\mathbf{p}|)J_3^{(j)}\}]_{\sigma',-j}=D_{\sigma,-j}^{(j)}[R(\hat{p})](|\mathbf{p}|/\kappa)^j\,,\quad(4.8)$$

$$D_{\sigma,j}^{(j)}[\mathcal{L}(\mathbf{p})]=\sum_{\sigma'}\bar{D}_{\sigma\sigma'}^{(j)}[R(\hat{p})][\exp\{\phi(|\mathbf{p}|)J_3^{(j)}\}]_{\sigma',j}=D_{\sigma,j}^{(j)}[R(\hat{p})](|\mathbf{p}|/\kappa)^j\,.\quad(4.9)$$

Note that the matrices $D^{(j)}[R]$ and $\bar{D}^{(j)}[R]$ for a pure rotation R are both equal, being given by the familiar $2j+1$-dimensional unitary representation[8] of the ordinary rotation group. [Note, also, that if we tried to construct a $(j,0)$ field for a right-handed particle, or a $(0,j)$ field for a left-handed particle, we would not only fail to get the desired Lorentz transformation property, but we would also find a catastrophic factor $|\mathbf{p}|^{-j}$ in the wave function.]

Using the wave functions (4.8) and (4.9) in (4.1) and (4.2), the annihilation fields now take the form

$$\varphi_\sigma^{(+)}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p[2|\mathbf{p}|]^{j-1/2}\times e^{ip\cdot x}D_{\sigma,-j}^{(j)}[R(\hat{p})]a(\mathbf{p},-j)\,,\quad(4.10)$$

$$\chi_\sigma^{(+)}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p[2|\mathbf{p}|]^{j-1/2}\times e^{ip\cdot x}D_{\sigma,j}^{(j)}[R(\hat{p})]a(\mathbf{p},j)\,.\quad(4.11)$$

We have redefined their normalization by replacing the factor κ^{-j} by 2^j. We see that only the ordinary unitary rotation matrices[8] are needed; $R(\hat{p})$ is the rotation that carries the z axis into the direction of $\mathbf{p}$.

If our particle has an antiparticle (perhaps itself), then there is available another operator $b^*(\mathbf{p},-\lambda)$ which transforms just like $a(\mathbf{p},\lambda)$ [see (2.43)], and which carries the same charge, baryon number, etc. It is then possible to define creation fields

$$\varphi_\sigma^{(-)}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p[2|\mathbf{p}|]^{j-1/2}\times e^{-ip\cdot x}D_{\sigma,-j}^{(j)}[R(\hat{p})]b^*(\mathbf{p},j)\,,\quad(4.12)$$

$$\chi_\sigma^{(-)}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p[2|\mathbf{p}|]^{j-1/2}\times e^{-ip\cdot x}D_{\sigma,j}^{(j)}[R(\hat{p})]b^*(\mathbf{p},-j)\,,\quad(4.13)$$

which satisfy (3.1), which transform according to (4.3) and (4.4), respectively, and which also transform like $\varphi^{(+)}$ and $\chi^{(+)}$ under gauge transformations of the first kind. [For a "purely neutral" particle,[6] $b^*(\mathbf{p},\lambda)$ is to be replaced by $a^*(\mathbf{p},\lambda)$.]

The most general fields satisfying all these conditions are linear combinations of creation and annihilation fields.

$$\varphi_\sigma(x)=\xi_L\varphi_\sigma^{(+)}(x)+\eta_R\varphi_\sigma^{(-)}(x)\,,\quad(4.14)$$

$$\chi_\sigma(x)=\xi_R\chi_\sigma^{(+)}(x)+\eta_L\chi_\sigma^{(-)}(x)\,.\quad(4.15)$$

They again transform as in (4.3) and (4.4):

$$U[\Lambda]\varphi_\sigma(x)U^{-1}[\Lambda]=\sum_{\sigma'}D_{\sigma\sigma'}^{(j)}[\Lambda^{-1}]\varphi_{\sigma'}(\Lambda x)\,,\quad(4.16)$$

$$U[\Lambda]\chi_\sigma(x)U^{-1}[\Lambda]=\sum_{\sigma'}\bar{D}_{\sigma\sigma'}^{(j)}[\Lambda]\chi_{\sigma'}(\Lambda x)\,.\quad(4.17)$$

[8] See, for example, M. E. Rose, *Elementary Theory of Angular Momentum* (J. Wiley & Sons, Inc., New York, 1957), p. 48 ff.

If these particles have no antiparticles (including themselves), then we have to take $\eta_L=\eta_R=0$. We will see in the next section that, instead, requirement (1.4) (and hence the Lorentz invariance of the S matrix) dictates full crossing symmetry, with $|\eta_R|=|\xi_L|$, $|\eta_L|=|\xi_R|$.

The fields obviously obey the Klein-Gordon equation

$$\Box^2\varphi_\sigma(x)=0;\quad \Box^2\chi_\sigma(x)=0. \tag{4.18}$$

However, they are $(2j+1)$-component objects constructed out of just two independent operators $a(\mathbf{p},\lambda)$, $b^*(\mathbf{p},-\lambda)$, and so they have a chance of obeying other field equations as well. It is not hard to see from (4.10)–(4.13) that they do indeed satisfy the additional field equations

$$[\mathbf{J}^{(j)}\cdot\nabla-j(\partial/\partial t)]\varphi(x)=0, \tag{4.19}$$

$$[\mathbf{J}^{(j)}\cdot\nabla+j(\partial/\partial t)]\chi(x)=0. \tag{4.20}$$

For $j=\frac{1}{2}$ these are the Weyl equations for the left- and right-handed neutrino fields, while for $j=1$ they are just Maxwell's free-space equations for left- and right-circularly polarized radiation:

$$\nabla\times[\mathbf{E}-i\mathbf{B}]+i(\partial/\partial t)[\mathbf{E}-i\mathbf{B}]=0, \tag{4.21}$$

$$\nabla\times[\mathbf{E}+i\mathbf{B}]-i(\partial/\partial t)[\mathbf{E}+i\mathbf{B}]=0. \tag{4.22}$$

The fact that these field equations are of first order for any spin seems to me to be of no great significance, since in the case of massive particles we can get along perfectly well with $(2j+1)$-component fields which satisfy only the Klein-Gordon equation.

V. CROSSING AND STATISTICS

We are assuming that the a's and b's satisfy the usual commutation (or anticommutation) rules (2.39), so it is easy to work out the commutators or anticommutators of the fields φ_σ and χ_σ defined by (4.10)–(4.15):

$$[\varphi_\sigma(x),\varphi_{\sigma'}{}^\dagger(y)]_\pm=\frac{1}{(2\pi)^3}\int\frac{d^3p}{2|\mathbf{p}|}\pi_{\sigma\sigma'}(p)\times[|\xi_L|^2e^{ip\cdot(x-y)}\pm|\eta_R|^2e^{-ip\cdot(x-y)}], \tag{5.1}$$

$$[\chi_\sigma(x),\chi_{\sigma'}{}^\dagger(y)]_\pm=\frac{1}{(2\pi)^3}\int\frac{d^3p}{2|\mathbf{p}|}\bar{\pi}_{\sigma\sigma'}(p)\times[|\xi_R|^2e^{ip\cdot(x-y)}\pm|\eta_L|^2e^{-ip\cdot(x-y)}], \tag{5.2}$$

where

$$\pi_{\sigma\sigma'}(p)=|2\mathbf{p}|^{2j}D_{\sigma,-j}{}^{(j)}[R(\hat{p})]D_{\sigma',-j}{}^{(j)*}[R(\hat{p})], \tag{5.3}$$

$$\bar{\pi}_{\sigma\sigma'}(p)=|2\mathbf{p}|^{2j}D_{\sigma,j}{}^{(j)}[R(\hat{p})]D_{\sigma',j}{}^{(j)*}[R(\hat{p})]. \tag{5.4}$$

These are the only nonvanishing commutators (or anticommutators) among the φ, $\varphi^\dagger$, χ, and $\chi^\dagger$ (except for a "purely neutral" particle, in which case χ is proportional to $\varphi^\dagger$; see Sec. IX).

The matrices π and $\bar{\pi}$ can be easily calculated by use of the obvious formulas

$$\delta_{\sigma,-j}\delta_{\sigma',-j}=\frac{1}{(2j)!}\Big[\prod_{\lambda=-j+1}^{j}(\lambda-J_3)\Big]_{\sigma\sigma'}, \tag{5.5}$$

$$\delta_{\sigma,j}\delta_{\sigma',j}=\frac{1}{(2j)!}\Big[\prod_{\lambda=-j}^{j-1}(J_3-\lambda)\Big]_{\sigma\sigma'}. \tag{5.6}$$

Applying the rotation matrix $D^{(j)}[R(\mathbf{p})]$ and multiplying by $|2\mathbf{p}|^{2j}$ gives

$$\pi(p)=\frac{2^{2j}}{(2j)!}\prod_{\lambda=-j+1}^{j}(\lambda p^0-\mathbf{p}\cdot\mathbf{J}), \tag{5.7}$$

$$\bar{\pi}(p)=\frac{2^{2j}}{(2j)!}\prod_{\lambda=-j}^{j-1}(\mathbf{p}\cdot\mathbf{J}-\lambda p^0). \tag{5.8}$$

These are monomials of order $2j$ in the light-like four-vector p^μ, so (5.1) and (5.2) now become

$$[\varphi_\sigma(x),\varphi_{\sigma'}{}^\dagger(y)]_\pm=\frac{1}{(2\pi)^3}\pi_{\sigma\sigma'}(-i\partial)\int\frac{d^3p}{2|\mathbf{p}|}\times[|\xi_L|^2e^{ip\cdot(x-y)}\pm(-)^{2j}|\eta_R|^2e^{-ip\cdot(x-y)}], \tag{5.9}$$

$$[\chi_\sigma(x),\chi_{\sigma'}{}^\dagger(y)]_\pm=\frac{1}{(2\pi)^3}\bar{\pi}_{\sigma\sigma'}(-i\partial)\int\frac{d^3p}{2|\mathbf{p}|}\times[|\xi_R|^2e^{ip\cdot(x-y)}\pm(-)^{2j}|\eta_L|^2e^{-ip\cdot(x-y)}]. \tag{5.10}$$

In order that (5.9) and (5.10) vanish for $x-y$ space-like, it is necessary and sufficient that $\exp[ip\cdot(x-y)]$ and $\exp[-ip\cdot(x-y)]$ have equal and opposite coefficients

$$|\xi_L|^2=\mp(-)^{2j}|\eta_R|^2, \tag{5.11}$$

$$|\xi_R|^2=\mp(-)^{2j}|\eta_L|^2. \tag{5.12}$$

So we must have the usual connection between spin and statistics

$$(\pm)=-(-)^{2j}, \tag{5.13}$$

and furthermore, every left- or right-handed particle must be associated, respectively, with a right- or left-handed antiparticle (perhaps itself) which enters into interactions with equal strength:

$$|\xi_L|=|\eta_R|,\quad |\xi_R|=|\eta_L|. \tag{5.14}$$

By redefining the phases of the a's and b's, and the normalization of φ and χ, we can therefore set

$$\xi_L=\eta_L=\xi_R=\eta_R=1 \tag{5.15}$$

with no loss of generality. The fields are now in their final form:

$$\varphi_\sigma(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p[2|\mathbf{p}|]^{j-1/2}D_{\sigma,-j}{}^{(j)}[R(\hat{p})]$$

$$\times[a(\mathbf{p},-j)e^{ip\cdot x}+b^*(\mathbf{p},j)e^{-ip\cdot x}],\quad(5.16)$$

$$\chi_\sigma(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p[2|\mathbf{p}|]^{j-1/2}D_{\sigma,j}{}^{(j)}[R(\hat{p})]$$

$$\times[a(\mathbf{p},j)e^{ip\cdot x}+b^*(\mathbf{p},-j)e^{-ip\cdot x}].\quad(5.17)$$

The commutator or anticommutators are

$$[\varphi_\sigma(x),\varphi_{\sigma'}{}^\dagger(y)]_\pm=i\pi_{\sigma\sigma'}(-i\partial)\Delta(x-y),\quad(5.18)$$

$$[\chi_\sigma(x),\chi_{\sigma'}{}^\dagger(y)]_\pm=i\bar{\pi}_{\sigma\sigma'}(-i\partial)\Delta(x-y),\quad(5.19)$$

where $i\Delta(x-y)$ is the commutator for zero mass and $j=0$.

$$\Delta(x)=\frac{-i}{(2\pi)^3}\int\frac{d^3p}{2|\mathbf{p}|}[e^{ip\cdot x}-e^{-ip\cdot x}]$$

$$=-(1/2\pi)\delta(x_\mu x^\mu)\epsilon(x).$$

If a particle has no additive quantum numbers like the photon, we must[6] set $b(\mathbf{p},\lambda)$ equal to $a(\mathbf{p},\lambda)$, and "causality" then tells us through (5.14) that the particle must exist in both left- and right-handed helicity states. Both fields $\varphi_\sigma(x)$ and $\chi_\sigma(x)$ can be constructed, and in fact we shall see in Sec. IX that φ is just proportional to $\chi^\dagger$.

On the other hand, a particle which carries some additive quantum number that distinguishes it from its antiparticle can possibly exist in only the left- *or* the right-handed helicity state, and "causality" only requires that it has an antiparticle of opposite helicity. (A familiar example is the neutrino.) In this case only one of the fields φ_σ and χ_σ can be constructed. Of course, if parity of charge conjugation are conserved, then both particle and antiparticle must exist in both left- and right-handed states, and both φ_σ and χ_σ exist.

VI. LORENTZ INVARIANCE

Our formulas (5.18) and (5.19) for the commutators or anticommutators were derived in a Lorentz invariant manner, but they do not look like invariant equations. It will be necessary to see how their invariance comes about before we are able to derive the Feynman rules.

It was shown in Appendix A of Ref. 1 that the familiar angular momentum matrices $J^{(j)}$ can be used to construct a pair of scalar $(2j+1)\times(2j+1)$ matrices Π and $\bar{\Pi}$, as monomials in a general four-vector q^μ:

$$\Pi_{\sigma\sigma'}(q)=(-)^{2j}t_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}}q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2j}},\quad(6.1)$$

$$\bar{\Pi}_{\sigma\sigma'}(q)=(-)^{2j}\bar{t}_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}}q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2j}},\quad(6.2)$$

with properties:

(a) Π and $\bar{\Pi}$ are scalars, in the sense that

$$D^{(j)}[\Lambda]\Pi(q)D^{(j)}[\Lambda]^\dagger=\Pi(\Lambda q),\quad(6.3)$$

$$\bar{D}^{(j)}[\Lambda]\bar{\Pi}(q)\bar{D}^{(j)}[\Lambda]^\dagger=\bar{\Pi}(\Lambda q).\quad(6.4)$$

(b) t and $\bar{t}$ are symmetric and traceless in $\mu_1\mu_2\cdots\mu_{2j}$.
(c) Π and $\bar{\Pi}$ are related by an inversion

$$\bar{\Pi}(-\mathbf{q},q^0)=\Pi(q).\quad(6.5)$$

(d) Π and $\bar{\Pi}^*$ are related by a similarity transformation

$$\bar{\Pi}^*(q)=C\Pi(q)C^{-1},\quad(6.6)$$

where

$$-\mathbf{J}^{(j)*}=C\mathbf{J}^{(j)}C^{-1}.\quad(6.7)$$

(e) Π and $\bar{\Pi}$ are further related by

$$\Pi(q)\bar{\Pi}(q)=\bar{\Pi}(q)\Pi(q)=(-q^2)^{2j}.\quad(6.8)$$

(f) If q is in the forward light cone then

$$\Pi(q)=(-q^2)^j\exp[-2\theta(q)\hat{q}\cdot\mathbf{J}^{(j)}],\quad(6.9)$$

$$\bar{\Pi}(q)=(-q^2)^j\exp[2\theta(q)\hat{q}\cdot\mathbf{J}^{(j)}],\quad(6.10)$$

$$\sinh\theta(q)\equiv[|\mathbf{q}|^2/-q^2]^{1/2}.\quad(6.11)$$

(g) For integer j and arbitrary q

$$\begin{aligned}\Pi^{(j)}(q)=&(-q^2)^j+[(-q^2)^{j-1}/2!](2\mathbf{q}\cdot\mathbf{J}^{(j)})(2\mathbf{q}\cdot\mathbf{J}^{(j)}-2q^0)\\&+[(-q^2)^{j-2}/4!](2\mathbf{q}\cdot\mathbf{J}^{(j)})[(2\mathbf{q}\cdot\mathbf{J}^{(j)})^2-(2\mathbf{q})^2]\\&\times[2\mathbf{q}\cdot\mathbf{J}^{(j)}-4q^0]+[(-q^2)^{j-3}/6!](2\mathbf{q}\cdot\mathbf{J}^{(j)})\\&\times[(2\mathbf{q}\cdot\mathbf{J}^{(j)})^2-(2\mathbf{q})^2][(2\mathbf{q}\cdot\mathbf{J}^{(j)})^2-(4\mathbf{q})^2]\\&\times[2\mathbf{q}\cdot\mathbf{J}^{(j)}-6q^0]+\cdots,\end{aligned}\quad(6.12)$$

the series cutting itself off automatically after $j+1$ terms.
(h) For half-integer j and arbitrary q

$$\begin{aligned}\Pi^{(j)}(q)=&(-q^2)^{j-1/2}[q^0-2\mathbf{q}\cdot\mathbf{J}^{(j)}]+(1/3!)(-q^2)^{j-3/2}\\&\times[(2\mathbf{q}\cdot\mathbf{J}^{(j)})^2-\mathbf{q}^2][3q^0-2\mathbf{q}\cdot\mathbf{J}^{(j)}]\\&+(1/5!)(-q^2)^{j-5/2}[(2\mathbf{q}\cdot\mathbf{J}^{(j)})^2-\mathbf{q}^2]\\&\times[(2\mathbf{q}\cdot\mathbf{J}^{(j)})^2-(3\mathbf{q})^2]\\&\times[5q^0-2\mathbf{q}\cdot\mathbf{J}^{(j)}]+\cdots,\end{aligned}\quad(6.13)$$

the series cutting itself off automatically after $j+\frac{1}{2}$ terms.

It follows from (6.12), (6.13), and (6.5) [or, more directly, from (6.9) and (6.10)] that for a light-like vector p^μ the monomials Π and $\bar{\Pi}$ simplify to

$$\Pi(p)=\frac{2^{2j}}{(2j)!}\prod_{\lambda=-j+1}^{j}(\lambda p^0-\mathbf{p}\cdot\mathbf{J}^{(j)}),\quad(6.14)$$

$$\bar{\Pi}(p)=\frac{2^{2j}}{(2j)!}\prod_{\lambda=-j+1}^{j}(\lambda p^0+\mathbf{p}\cdot\mathbf{J}^{(j)}),\quad(6.15)$$

or in terms of the matrices (5.7), (5.8)

$$\Pi_{\sigma\sigma'}(p)=\pi_{\sigma\sigma'}(p)\quad[p\text{ light-like}],\quad(6.16)$$

$$\bar{\Pi}_{\sigma\sigma'}(p)=\bar{\pi}_{\sigma\sigma'}(p)\quad[p\text{ light-like}].\quad(6.17)$$

The Lorentz invariance of formulas (5.18) and (5.19) for the commutators or anticommutators now follows immediately from (6.3) and (6.4).

VII. THE FEYNMAN RULES

The Hamiltonian density $\mathcal{H}(x)$ is to be constructed as an invariant polynomial in the $(2j+1)$-component fields $\varphi_\sigma(x)$ and $\chi_\sigma(x)$, without any distinction made between zero and nonzero mass. In each term of $\mathcal{H}(x)$ all σ indices on the $\varphi_\sigma(x)$ are to be coupled together to form a scalar, using Clebsch-Gordan coefficients in the familiar way. The same is to be done *independently* with the indices on the $\chi_\sigma(x)$. If adjoint fields enter in $\mathcal{H}(x)$ then $C_{\sigma\sigma'}{}^{-1}\chi_{\sigma'}{}^\dagger(x)$ is to be treated like $\varphi_\sigma(x)$ and $C_{\sigma\sigma'}{}^{-1}\varphi_{\sigma'}(x)^\dagger$ is to be treated like $\chi_\sigma(x)$; the matrix C is defined by

$$\bar{D}^{(j)}[\Lambda]^* = CD^{(j)}[\Lambda]C^{-1}, \tag{7.1}$$

or more specifically,

$$-\mathbf{J}^{(j)*} = C\mathbf{J}^{(j)}C^{-1}. \tag{7.2}$$

[We use an asterisk for the ordinary complex conjugate of a matrix.] If derivatives appear they will enter as a 2×2 matrix:

$$\partial_{\sigma\sigma'} = \sigma_{\sigma\sigma'}{}^i(\partial/\partial x^i) - \delta_{\sigma\sigma'}(\partial/\partial t), \tag{7.3}$$

where σ^i are the usual Pauli spin matrices; the indices σ and σ' are to be treated as if they appeared respectively on $j=\frac{1}{2}$ fields φ_σ and $\chi_{\sigma'}$.

We list below some typical examples of possible invariant terms in $\mathcal{H}(x)$:

$$\sum_{\sigma_1\sigma_2\sigma_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} \varphi_{\sigma_1}{}^{(j_1)}(x)\varphi_{\sigma_2}{}^{(j_2)}(x)\varphi_{\sigma_3}{}^{(j_3)}(x), \tag{7.4}$$

$$\sum_{\sigma_1\sigma_2\sigma_3\sigma_3'} \begin{pmatrix} j_1 & j_2 & j_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} \times \varphi_{\sigma_1}{}^{(j_1)}(x)\varphi_{\sigma_2}{}^{(j_2)}(x) C_{\sigma_3\sigma_3'}{}^{-1}\chi_{\sigma_3'}{}^{(j_3)}(x)^\dagger, \tag{7.5}$$

$$\sum_{\sigma_1\sigma_2\sigma_3\sigma\sigma'} \begin{pmatrix} j_1 & j_2 & \frac{1}{2} \\ \sigma_1 & \sigma_2 & \sigma \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \sigma' & \sigma_3 & 0 \end{pmatrix} \times \varphi_{\sigma_1}{}^{(j_1)}(x)\varphi_{\sigma_2}{}^{(j_2)}(x)\partial_{\sigma\sigma'}\chi_{\sigma_3}{}^{(\frac{1}{2})}(x), \tag{7.6}$$

etc. The fields φ_σ and χ_σ appearing here may be either of zero or of nonzero mass.

The S matrix can be calculated from $\mathcal{H}(x)$ by using Wick's theorem to derive the Feynman rules, as we did in Sec. V of Ref. 1. The only additional information needed here is a statement of the wave functions corresponding to external mass zero lines, and a formula for the propagators corresponding to internal mass zero lines.

The factor arising from the destruction or creation at x of a massless particle or antiparticle of helicity $\lambda=\pm j$ can be determined from (5.16) and (5.17) as the coefficient of the appropriate creation or annihilation operator in φ_σ, χ_σ, $\varphi_\sigma{}^\dagger$, or $\chi_\sigma{}^\dagger$:

$$(2\pi)^{-3/2}(2|\mathbf{p}|)^{j-1/2}D_{\sigma,\lambda}{}^{(j)}[R(\hat{p})]e^{ip\cdot x} \quad \text{[particle destroyed]}, \tag{7.7}$$

$$(2\pi)^{-3/2}(2|\mathbf{p}|)^{j-1/2}D_{\sigma,\lambda}{}^{(j)}[R(\hat{p})]^* e^{-ip\cdot x} \quad \text{[particle created]}, \tag{7.8}$$

$$(2\pi)^{-3/2}(2|\mathbf{p}|)^{j-1/2}D_{\sigma,-\lambda}{}^{(j)}[R(\hat{p})]e^{-ip\cdot x} \quad \text{[antiparticle created]}, \tag{7.9}$$

$$(2\pi)^{-3/2}(2|\mathbf{p}|)^{j-1/2}D_{\sigma,-\lambda}{}^{(j)}[R(\hat{p})]^* e^{ip\cdot x} \quad \text{[antiparticle destroyed]}. \tag{7.10}$$

We remind the reader that $D^{(j)}[R]$ is the usual $(2j+1)\times(2j+1)$ unitary matrix[8] corresponding to an ordinary rotation R, and that $R(\hat{p})$ is the rotation that carries the z axis into the direction of $\mathbf{p}$.

The "raw" propagator corresponding to an internal massless particle line running from x to y is

$$\langle T\{\varphi_\sigma(x),\varphi_{\sigma'}{}^\dagger(y)\}\rangle_0 = \theta(x-y)\langle\varphi_\sigma(x)\varphi_{\sigma'}{}^\dagger(y)\rangle_0 + (-)^{2j}\theta(y-x)\langle\varphi_{\sigma'}{}^\dagger(y)\varphi_\sigma(x)\rangle_0, \tag{7.11}$$

or

$$\langle T\{\chi_\sigma(x),\chi_{\sigma'}{}^\dagger(y)\}\rangle_0 = \theta(x-y)\langle\chi_\sigma(x)\chi_{\sigma'}{}^\dagger(y)\rangle_0 + (-)^{2j}\theta(y-x)\langle\chi_{\sigma'}{}^\dagger(y)\chi_\sigma(x)\rangle_0. \tag{7.12}$$

An elementary calculation using (5.16), (5.17), (5.3), (5.4), (6.16), and (6.17) gives the vacuum expectation values as

$$\langle\varphi_\sigma(x)\varphi_{\sigma'}{}^\dagger(y)\rangle_0 = i\Pi_{\sigma\sigma'}(-i\partial)\Delta_+(x-y), \tag{7.13}$$

$$(-)^{2j}\langle\varphi_{\sigma'}{}^\dagger(y)\varphi_\sigma(x)\rangle_0 = i\Pi_{\sigma\sigma'}(-i\partial)\Delta_+(y-x), \tag{7.14}$$

and

$$\langle\chi_\sigma(x)\chi_{\sigma'}{}^\dagger(y)\rangle_0 = i\bar{\Pi}_{\sigma\sigma'}(-i\partial)\Delta_+(x-y), \tag{7.15}$$

$$(-)^{2j}\langle\chi_{\sigma'}{}^\dagger(y)\chi_\sigma(x)\rangle_0 = i\bar{\Pi}_{\sigma\sigma'}(-i\partial)\Delta_+(y-x), \tag{7.16}$$

where

$$i\Delta_+(x) \equiv \frac{1}{(2\pi)^3}\int \frac{d^3p}{2|\mathbf{p}|}e^{ip\cdot x} = \frac{1}{4\pi^2}\left[\frac{1}{x^2} - i\pi\delta(x^2)\epsilon(x)\right]. \tag{7.17}$$

As discussed in Ref. 1, the presence of the θ functions in (7.11) and (7.12) makes these propagators noncovariant at the point $x=y$, for spins $j\geqq 1$. In order that the S matrix be Lorentz invariant, it is necessary to assume that noncovariant contact interactions appear in $\mathcal{H}(x)$ which cancel the noncovariant terms in (7.11) and (7.12). (The Coulomb interaction in Coulomb gauge is such a contact interaction, made necessary by the unit spin rather than by the zero mass of the photon.) With this understanding, we can move the derivative operators $\Pi(-i\partial)$ and $\bar{\Pi}(-i\partial)$ in (7.13)–

(7.16) to the left of the θ functions in (7.11) and (7.12), obtaining the propagators

$$S_{\sigma\sigma'}(x-y)=-i\Pi_{\sigma\sigma'}(-i\partial)\Delta^c(x-y)$$
$$=-i^{2j+1}t_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}}\partial_{\mu_1}\partial_{\mu_2}\cdots$$
$$\times\partial_{\mu_{2j}}\Delta^c(x-y)\,, \quad (7.18)$$

$$\bar{S}_{\sigma\sigma'}(x-y)=-i\bar{\Pi}_{\sigma\sigma'}(-i\partial)\Delta^c(x-y)$$
$$=-i^{2j+1}\bar{t}_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}}\partial_{\mu_1}\partial_{\mu_2}\cdots$$
$$\times\partial_{\mu_{2j}}\Delta^c(x-y)\,, \quad (7.19)$$

where $-i\Delta^c(x-y)$ is the usual propagator for spin zero and mass zero

$$-i\Delta^c(x)=i\theta(x)\Delta_+(x)+i\theta(-x)\Delta_+(-x)$$
$$=+[1/4\pi^2(x^2+i\epsilon)]. \quad (7.20)$$

Equations (6.3) and (6.4) show that these propagators are covariant in the sense that

$$D^{(j)}[\Lambda]S(x)D^{(j)}[\Lambda]^\dagger=S(\Lambda x)\,, \quad (7.21)$$

$$\bar{D}^{(j)}[\Lambda]\bar{S}(x)\bar{D}^{(j)}[\Lambda]^\dagger=\bar{S}(\Lambda x)\,. \quad (7.22)$$

The propagators in momentum space are given by the Fourier transforms of (7.18) and (7.19)

$$S(q)=\int d^4x e^{iq\cdot x}S(x)=-i\Pi(q)/q^2-i\epsilon\,, \quad (7.23)$$

$$\bar{S}(q)=\int d^4x e^{-iq\cdot x}\bar{S}(x)=-i\bar{\Pi}(q)/q^2-i\epsilon. \quad (7.24)$$

Explicit formulas for the monomials $\Pi(q)$ and $\bar{\Pi}(q)$ are given in Eqs. (6.12), (6.13), and (6.5), or for $j\leqq 3$ in Table I of Ref. 1.

VIII. GENERAL HELICITY AMPLITUDES AND THE LIMIT $m\to 0$

The Feynman rules were given in Ref. 1 for incoming and outgoing massive particles having prescribed values for the z components of their spins. It turns out, however, that the external-line wave functions are much simpler in the Jacob-Wick formalism,[4] where initial and final states are labeled instead by the particle helicities. For $m=0$, of course, we have had no choice, since only the helicity amplitudes are physically meaningful. We will first derive the helicity wave functions for $m>0$, and then use them to show how the Feynman rules given here for $m=0$ can be obtained by taking the limit $m\to 0$ of the Feynman rules for positive m.

According to the Feynman rules of Ref. 1, the wave function for a particle of spin j, $J_z=\mu$, momentum $\mathbf{p}$, and mass m, destroyed by $\varphi_\sigma(x)$, is

$$u_\sigma(x;\mathbf{p},\mu)=(2\omega)^{-1/2}(2\pi)^{-3/2}$$
$$\times[\exp(-\hat{\mathbf{p}}\cdot\mathbf{J}^{(j)}\theta)]_{\sigma\mu}e^{ip\cdot x}, \quad (8.1)$$

where

$$\omega=[\mathbf{p}^2+m^2]^{1/2},\qquad \sinh\theta=|\mathbf{p}|/m. \quad (8.2)$$

(It should be kept in mind that the index σ, which is of no direct physical significance, will appear on some other wave function or propagator, and eventually be summed over.) The corresponding wave function for a particle of definite helicity λ is

$$U_\sigma(x;\mathbf{p},\lambda)=\sum_\mu D_{\mu\lambda}{}^{(j)}[R(\hat{p})]u_\sigma(x;\mathbf{p},\mu)\,, \quad (8.3)$$

where $R(\hat{p})$, as always, is the rotation that carries the z axis into the direction of $\mathbf{p}$. Using (8.1) in (8.3) gives

$$U_\sigma(x;\mathbf{p},\lambda)=(2\omega)^{-1/2}(2\pi)^{-3/2}$$
$$\times\{\exp(-\hat{p}\cdot\mathbf{J}^{(j)}\theta)D^{(j)}[R(\hat{p})]\}_{\sigma\lambda}e^{ip\cdot x}$$
$$=(2\omega)^{-1/2}(2\pi)^{-3/2}$$
$$\times\{D^{(j)}[R(\hat{p})]\exp(-J_3{}^{(j)}\theta)\}_{\sigma\lambda}e^{ip\cdot x}$$
$$=(2\omega)^{-1/2}(2\pi)^{-3/2}D_{\sigma\lambda}{}^{(j)}[R(\hat{p})]e^{-\lambda\theta}e^{ip\cdot x}. \quad (8.4)$$

Furthermore we see from (8.2) that

$$e^{-\lambda\theta}=[\omega(\mathbf{p})+|\mathbf{p}|/m]^{-\lambda}. \quad (8.5)$$

In order to avoid m's appearing in the denominator of U_σ for negative helicity, it will be convenient to renormalize all fields of mass m by multiplying them with a factor m^j. With this understanding, the wave function for a particle of spin j, helicity λ, momentum $\mathbf{p}$, and mass m, destroyed by $\varphi_\sigma(x)$, is

$$U_\sigma(x;\mathbf{p},\lambda)=(2\omega)^{-1/2}(2\pi)^{-3/2}D_{\sigma\lambda}{}^{(j)}[R(\hat{p})]$$
$$\times m^{j+\lambda}(\omega+|\mathbf{p}|)^{-\lambda}e^{ip\cdot x}. \quad (8.6)$$

The wave function for the creation of the same particle by $\varphi_\sigma{}^\dagger(x)$ is just the complex conjugate

$$U_\sigma{}^*(x;\mathbf{p},\lambda)=(2\omega)^{-1/2}(2\pi)^{-3/2}D_{\sigma\lambda}{}^{(j)*}[R(\hat{p})]$$
$$\times m^{j+\lambda}(\omega+|\mathbf{p}|)^{-\lambda}e^{-ip\cdot x}. \quad (8.7)$$

The wave function for the creation by $\varphi_\sigma(x)$ of the antiparticle with helicity λ and spin $\mathbf{p}$ can be easily obtained in the same way from Eq. (5.4) of Ref. 1, by using the relations

$$D^{(j)*}[R(\hat{p})]=CD^{(j)}[R(\hat{p})]C^{-1},$$
$$C_{\mu\lambda}{}^{-1}=(-)^{-j+\lambda}\delta_{\mu,-\lambda}.$$

We find that the antiparticle creation wave function is

$$V_\sigma(x;\mathbf{p},\lambda)=(2\omega)^{-1/2}(2\pi)^{-3/2}(-)^{-j+\lambda}D_{\sigma,-\lambda}{}^{(j)}[R(\hat{p})]$$
$$\times m^{j-\lambda}(\omega+|\mathbf{p}|)^{\lambda}e^{-ip\cdot x}, \quad (8.8)$$

and the wave function for destruction of the same antiparticle by $\varphi_\sigma{}^\dagger(x)$ is the complex conjugate

$$V_\sigma{}^*(x;\mathbf{p},\lambda)=(2\omega)^{-1/2}(2\pi)^{-3/2}(-)^{-j+\lambda}D_{\sigma,-\lambda}{}^{(j)*}[R(\hat{p})]$$
$$\times m^{j-\lambda}(\omega+|\mathbf{p}|)^{\lambda}e^{+ip\cdot x}. \quad (8.9)$$

A massive particle can be created or destroyed in any helicity state by either the $(j,0)$ field $\varphi_\sigma(x)$ or the $(0,j)$ field $\chi_\sigma(x)$. Inspection of the field $\chi_\sigma(x)$ given in Eq. (6.9) of Ref. 1 shows that the wave functions corresponding to (8.6)–(8.9) are given by replacing θ by $-\theta$, and supplying a sign $(-)^{2j}$ for antiparticles:

$$\bar{U}_\sigma(x;\mathbf{p},\lambda)=(2\omega)^{-1/2}(2\pi)^{-3/2}D_{\sigma\lambda}{}^{(j)}[R(\hat{p})]\times m^{j-\lambda}(\omega+|\mathbf{p}|)^\lambda e^{ip\cdot x}\quad\text{[particle destroyed]},\tag{8.10}$$

$$\bar{U}_\sigma{}^*(x;\mathbf{p},\lambda)=(2\omega)^{-1/2}(2\pi)^{-3/2}D_{\sigma\lambda}{}^{(j)*}[R(\hat{p})]\times m^{j-\lambda}(\omega+|\mathbf{p}|)^\lambda e^{-ip\cdot x}\quad\text{[particle created]},\tag{8.11}$$

$$\bar{V}_\sigma(x;p\lambda)=(2\omega)^{-1/2}(2\pi)^{-3/2}(-)^{j+\lambda}D_{\sigma,-\lambda}{}^{(j)}[R(\hat{p})]\times m^{j+\lambda}(\omega+|\mathbf{p}|)^{-\lambda}e^{-ip\cdot x}\quad\text{[antiparticle created]},\tag{8.12}$$

$$\bar{V}_\sigma{}^*(x;\mathbf{p},\lambda)=(2\omega)^{-1/2}(2\pi)^{-3/2}(-)^{j+\lambda}D_{\sigma,-\lambda}{}^{(j)*}[R(\hat{p})]\times m^{j+\lambda}(\omega+|\mathbf{p}|)^{-\lambda}e^{+ip\cdot x}\quad\text{[antiparticle destroyed]}.\tag{8.13}$$

Now suppose that $m\to 0$, or, more precisely, that $|\mathbf{p}|/m\to\infty$. The only wave functions among (8.6)–(8.13) that survive in this limit are (8.6), (8.7), (8.12), and (8.13) for $\lambda=-j$, and (8.8), (8.9), (8.10), and (8.11) for $\lambda=+j$. This agrees with the situation for $m=0$, in which case we know that φ_σ and $\varphi_\sigma{}^\dagger$ can only create and destroy particles with $\lambda=-j$ and antiparticles with $\lambda=+j$, while χ_σ and $\chi_\sigma{}^\dagger$ only create and destroy particles with $\lambda=+j$ and antiparticles with $\lambda=-j$. Furthermore, if we set $\lambda=-j$ in (8.6) or $\lambda=+j$ in (8.10) we see that these wave functions reduce for $|\mathbf{p}|/m\to\infty$ to the particle destruction wave function given for $m=0$ by (7.7). The same agreement is obtained on comparison of (8.7) and (8.11) with (7.8), (8.8), and (8.12) with (7.9), and (8.9), and (8.13) with (7.10). [The observation that particles described only by $\varphi_\sigma(x)$ are difficult to create or destroy for $|\mathbf{p}|\gg m$ in any helicity state other than $\lambda=-j$ is very familiar for electrons in beta decay.]

The propagators for an internal φ or χ line are given in Ref. 1 as

$$S_{\sigma\sigma'}(x-y)=-i\Pi_{\sigma\sigma'}(-i\partial)\Delta^c(x-y;m),\tag{8.14}$$

$$\bar{S}_{\sigma\sigma'}(x-y)=-i\bar{\Pi}_{\sigma\sigma'}(-i\partial)\Delta^c(x-y;m).\tag{8.15}$$

[Recall that we are now using fields renormalized by a factor m^j, so the factor m^{-2j} in Eq. (5.7) of Ref. 1 is absent here.] We see that the propagators given for $m=0$ by (7.18) and (7.19) are the limits respectively of (8.14) and (8.15) as $m\to 0$. For $m\neq 0$ there is also a "transition propagator" between φ_σ and $\chi_{\sigma'}{}^\dagger$, but it is proportional to m^{2j} and disappears as $m\to 0$.

In contrast, the Feynman rules for $m=0$ could *not* be obtained as the limit as $m\to 0$ of the corresponding rules for $m>0$, if we used one of the field types like $(j/2,j/2)$ which are forbidden by the theorem of Sec. III. For example, it is well known that the propagator for a vector field has a longitudinal part which blows up as m^{-2} for $m\to 0$; this is just our punishment for attempting to use the forbidden $(\frac{1}{2},\frac{1}{2})$ field type for $j=1$ particles of zero mass.[3]

IX. T, C, AND P

Time-reversal (**T**) and space inversion (**P**) are classically defined as transforming a particle of momentum **p** and helicity λ into

$$\mathbf{T}|\mathbf{p},\lambda\rangle\propto|-\mathbf{p},\lambda\rangle,\tag{9.1}$$

$$\mathbf{P}|\mathbf{p},\lambda\rangle\propto|-\mathbf{p},-\lambda\rangle,\tag{9.2}$$

while charge conjugation (**C**) just changes all particles into antiparticles, with no change in **p** and λ. However, in quantum mechanics there appear phases in (9.1) and (9.2), which we shall see are necessarily *momentum-dependent* for massless particles. In order to get these phases right it is necessary first to define the action of **T** and **P** on our standard states $|\lambda\rangle$ of momentum $k=\{0,0,\kappa\}$, and then use the definition (2.31) of $|\mathbf{p},\lambda\rangle$.

We will define "standard phases" $\eta_\lambda(\mathbf{T})$ and $\eta_\lambda(\mathbf{P})$ by

$$\mathbf{T}|\lambda\rangle=\eta_\lambda{}^*(\mathbf{T})U[R_c]|\lambda\rangle,\tag{9.3}$$

$$\mathbf{P}|\lambda\rangle=(-)^{j+\lambda}\eta_\lambda{}^*(\mathbf{P})U[R_c]|-\lambda\rangle,\tag{9.4}$$

where R_c is some fixed but arbitrary rotation such that

$$R_c\{0,0,1\}=\{0,0,-1\},\tag{9.5}$$

so that $U[R_c]|\lambda\rangle$ is a state of momentum $\{0,0,-\kappa\}$. [The factor $(-)^{j+\lambda}$ is extracted from $\eta_\lambda{}^*(\mathbf{P})$ for convenience later.] In order to calculate the effect of **T** and **P** on $|\mathbf{p},\lambda\rangle$ we need the well-known formulas

$$\mathbf{T}J_i\mathbf{T}^{-1}=-J_i,\tag{9.6}$$

$$\mathbf{T}K_i\mathbf{T}^{-1}=K_i,\tag{9.7}$$

$$\mathbf{P}J_i\mathbf{P}^{-1}=J_i,\tag{9.8}$$

$$\mathbf{P}K_i\mathbf{P}^{-1}=-K_i.\tag{9.9}$$

[It is easy to check that (9.6)–(9.9) are consistent with the commutation relations (2.16)–(2.18), if we recall that **T** is antiunitary.] According to (2.31) and (4.7), the state $|\mathbf{p},\lambda\rangle$ is

$$|\mathbf{p},\lambda\rangle\equiv[\kappa/|\mathbf{p}|]^{1/2}U[R(\hat{p})]\exp[-i\phi(|\mathbf{p}|)K_3]|\lambda\rangle,\tag{9.10}$$

so therefore

$$\mathbf{T}|\mathbf{p},\lambda\rangle=\eta_\lambda{}^*(\mathbf{T})[\kappa/|\mathbf{p}|]^{1/2}U[R(\hat{p})]\times\exp[i\phi(|\mathbf{p}|)K_3]U[R_c]|\lambda\rangle,$$

$$\mathbf{P}|\mathbf{p},\lambda\rangle=(-)^{j+\lambda}\eta_\lambda{}^*(\mathbf{P})[\kappa/|\mathbf{p}|]^{1/2}U[R(\hat{p})]\times\exp[i\phi(|\mathbf{p}|)K_3]U[R_c]|-\lambda\rangle.$$

But

$$U^{-1}[R_c]K_3U[R_c]=-K_3,$$

and thus

$$\mathbf{T}|\mathbf{p},\lambda\rangle=\eta_\lambda{}^*(\mathbf{T})[\kappa/|\mathbf{p}|]^{1/2}U[R(\hat{p})R_c]\times\exp[-i\phi(|\mathbf{p}|)K_3]|\lambda\rangle,\tag{9.11}$$

$$\mathbf{P}|\mathbf{p},\lambda\rangle=(-)^{j+\lambda}\eta_\lambda{}^*(\mathbf{P})[\kappa/|\mathbf{p}|]^{1/2}U[R(\hat{p})R_c]\times\exp[-i\phi(|\mathbf{p}|)K_3]|-\lambda\rangle.\tag{9.12}$$

The rotation $R(\hat{p})R_c$ carries the z axis into the direction of $-\mathbf{p}$, and must therefore be the product of $R(-\hat{p})$

times a rotation of $\Phi(\hat{p})$ degrees about the z axis

$$U[R(\hat{p})R_c]=U[R(-\hat{p})]\exp[i\Phi(\hat{p})J_3]. \quad (9.13)$$

The angle $\Phi(\hat{p})$ depends on how we standardize R_c and $R(\hat{p})$, but we will fortunately not need to calculate it, as it will cancel in the field transformation laws. Using (9.13) in (9.11) and (9.12), and recalling that J_3 commutes with K_3, we have at last

$$\mathbf{T}|\mathbf{p},\lambda\rangle=\eta_\lambda^*(\mathbf{T})\exp[i\lambda\Phi(\hat{p})]|-\mathbf{p},\lambda\rangle, \quad (9.14)$$

$$\mathbf{P}|\mathbf{p},\lambda\rangle=(-)^{j+\lambda}\eta_\lambda^*(\mathbf{P})\exp[-i\lambda\Phi(\hat{p})]|-\mathbf{p},-\lambda\rangle. \quad (9.15)$$

These one-particle transformation equations can be translated immediately into transformation rules for the annihilation operator:

$$\mathbf{T}a(\mathbf{p},\lambda)\mathbf{T}^{-1}=\eta_\lambda(\mathbf{T})\exp[-i\lambda\Phi(\hat{p})]a(-\mathbf{p},\lambda), \quad (9.16)$$

$$\mathbf{P}a(\mathbf{p},\lambda)\mathbf{P}^{-1}=(-)^{j+\lambda}\eta_\lambda(\mathbf{P}) \times\exp[i\lambda\Phi(\hat{p})]a(-\mathbf{p},-\lambda). \quad (9.17)$$

The antiparticle operators will transform similarly, but perhaps with different "standard" phases $\bar{\eta}_\lambda(\mathbf{T})$ and $\bar{\eta}_\lambda(\mathbf{p})$:

$$\mathbf{T}b(\mathbf{p},\lambda)\mathbf{T}^{-1}=\bar{\eta}_\lambda(\mathbf{T})\exp[-i\lambda\Phi(\hat{p})]b(-\mathbf{p},\lambda), \quad (9.18)$$

$$\mathbf{P}b(\mathbf{p},\lambda)\mathbf{P}^{-1}=(-)^{j+\lambda}\bar{\eta}_\lambda(\mathbf{P}) \times\exp[i\lambda\Phi(\hat{p})]b(-\mathbf{p},-\lambda). \quad (9.19)$$

And, of course, **C** just changes a's into b's and vice versa.

$$\mathbf{C}a(\mathbf{p},\lambda)\mathbf{C}^{-1}=\eta_\lambda(\mathbf{C})b(\mathbf{p},\lambda), \quad (9.20)$$

$$\mathbf{C}b(\mathbf{p},\lambda)\mathbf{C}^{-1}=\bar{\eta}_\lambda(\mathbf{C})a(\mathbf{p},\lambda). \quad (9.21)$$

The phases $\eta_\lambda(\mathbf{T},\mathbf{C},\mathbf{P})$, $\bar{\eta}_\lambda(\mathbf{T},\mathbf{C},\mathbf{P})$ are partly arbitrary,[9] partly determined by the structure of the Hamiltonian, and partly fixed by the specifically field-theoretic considerations below.

In order to calculate the effect of **T**, **C**, and **P** on the fields $\varphi_\sigma(x)$ and $\chi_\sigma(x)$, it will be necessary to use the well-known reality property of the rotation matrices

$$D^{(j)}[R]^*=CD^{(j)}[R]C^{-1}, \quad (9.22)$$

where, with the usual phase conventions,

$$C_{\sigma'\sigma}=(-)^{j+\sigma}\delta_{\sigma',-\sigma}=[\exp(i\pi J_2^{(j)})]_{\sigma'\sigma}. \quad (9.23)$$

We shall fix the rotation R_c introduced in Eq. (9.5) as a rotation of 180° about the y axis, such that

$$D^{(j)}[R_c]=C^{-1}=(-)^{2j}C. \quad (9.24)$$

Another needed relation then follows from (9.13).

$$D_{\sigma\lambda}^{(j)}[R(\hat{p})] =(-)^{j+\lambda}\exp[-i\lambda\Phi(\hat{p})]D_{\sigma,-\lambda}^{(j)}[R(-\hat{p})]. \quad (9.25)$$

The effect of **T**, **C**, and **P** on the fields (5.16) and (5.17) can now be easily determined by using (9.16)–(9.25):

$$\mathbf{T}\varphi_\sigma(x)\mathbf{T}^{-1}=\eta_{-j}(\mathbf{T})\sum_{\sigma'}C_{\sigma\sigma'}\varphi_{\sigma'}(\mathbf{x},-x^0), \quad (9.26)$$

$$\mathbf{T}\chi_\sigma(x)\mathbf{T}^{-1}=\eta_{j}(\mathbf{T})\sum_{\sigma'}C_{\sigma\sigma'}\chi_{\sigma'}(\mathbf{x},-x^0), \quad (9.27)$$

$$\mathbf{C}\varphi_\sigma(x)C^{-1}=\eta_{-j}(\mathbf{C})\sum_{\sigma'}C_{\sigma\sigma'}^{-1}\chi_{\sigma'}^\dagger(x), \quad (9.28)$$

$$\mathbf{C}\chi_\sigma(x)\mathbf{C}^{-1}=\eta_{j}(\mathbf{C})(-)^{2j}\sum_{\sigma'}C_{\sigma\sigma'}^{-1}\varphi_{\sigma'}^\dagger(x), \quad (9.29)$$

$$\mathbf{P}\varphi_\sigma(x)\mathbf{P}^{-1}=\eta_{-j}(\mathbf{P})\chi_\sigma(-\mathbf{x},x^0), \quad (9.30)$$

$$\mathbf{P}\chi_\sigma(x)\mathbf{P}^{-1}=\eta_{j}(\mathbf{P})\varphi_\sigma(-\mathbf{x},x^0). \quad (9.31)$$

In deriving (9.26)–(9.31) it is necessary to fix the antiparticle inversion phases as

$$\bar{\eta}_\lambda(\mathbf{T})=\eta_{-\lambda}^*(\mathbf{T}), \quad (9.32)$$

$$\bar{\eta}_\lambda(\mathbf{C})=\eta_{-\lambda}^*(\mathbf{C}), \quad (9.33)$$

$$\bar{\eta}_\lambda(\mathbf{P})=(-)^{2j}\eta_{-\lambda}^*(\mathbf{P}), \quad (9.34)$$

because any other choice of the $\bar{\eta}_\lambda$ would result in the creation and annihilation parts of the field transforming with different phases, and would therefore destroy the possibility of simple transformation laws.

It is interesting that the transformation rules (9.26)–(9.31) turn out to be identical with those derived in Sec. 6 of Ref. 1 for the case of massive particles, though the derivation has been different in many respects. The same is true of the phase relations (9.32)–(9.34), except that the only correlated particle and antiparticle inversion phases are those of opposite helicity. In particular, (9.34) tells us that a left- or right-handed particle plus a right- or left-handed antiparticle together have intrinsic parity

$$\eta_{-\lambda}(\mathbf{P})\bar{\eta}_\lambda(\mathbf{P})=(-)^{2j}, \quad (9.35)$$

while the intrinsic parity of a massless particle antiparticle pair of the same helicity is not fixed by these general field-theoretic arguments.

If a particle is its own antiparticle[6] then we must set

$$b(\mathbf{p},\lambda)=a(\mathbf{p},\lambda). \quad (9.36)$$

In this special case, the $(j,0)$ and $(0,j)$ fields are related by

$$\chi_\sigma^\dagger(x)=\sum_{\sigma'}C_{\sigma\sigma'}\varphi_{\sigma'}(x), \quad (9.37)$$

$$\varphi_\sigma^\dagger(x)=(-)^{2j}\sum_{\sigma'}C_{\sigma\sigma'}\chi_{\sigma'}(x). \quad (9.38)$$

Also (9.36) requires that the antiparticle inversion phases $\bar{\eta}_\lambda$ be equal to the corresponding η_λ, and therefore (9.32)–(9.34) provide relations between η_λ and $\eta_{-\lambda}$:

$$\eta_\lambda(\mathbf{T})=\eta_{-\lambda}^*(\mathbf{T}), \quad (9.39)$$

$$\eta_\lambda(\mathbf{C})=\eta_{-\lambda}^*(\mathbf{C}), \quad (9.40)$$

$$\eta_\lambda(\mathbf{P})=(-)^{2j}\eta_{-\lambda}^*(\mathbf{P}). \quad (9.41)$$

[9] For a general discussion, see G. Feinberg and S. Weinberg, Nuovo Cimento 14, 571 (1959).

However, there is still no necessity for any of these phases to be real.

Observe that (9.17) and (9.19)–(9.21) make sense only if both the particle and its antiparticle each exist in both helicity states $\lambda=\pm j$. For a particle not identical with its antiparticle, this is now a part of the assumption of **C** or **P** invariance, whereas in the case of massive particles it followed directly from the Lorentz invariance of the S matrix.

In contrast, **T** conservation leaves open the possibility that the particle exists in only one of the two helicity states, with an antiparticle of the opposite helicity. This is consistent with (9.26) and (9.27), which show that **T** does not mix φ_σ and χ_σ. The same is true of the combined inversion **CP**.

$$\mathbf{CP}\varphi_\sigma(x)\mathbf{P}^{-1}\mathbf{C}^{-1}=\eta_j(\mathbf{C})\eta_{-j}(\mathbf{P})\sum_{\sigma'} C_{\sigma\sigma'}{}^{-1}\varphi_{\sigma'}{}^\dagger(-\mathbf{x},x^0)\,,\quad(9.42)$$

$$\mathbf{CP}\chi_\sigma(x)\mathbf{P}^{-1}\mathbf{C}^{-1}=\eta_{-j}(\mathbf{C})\eta_j(\mathbf{P})\sum_{\sigma'} C_{\sigma\sigma'}{}^{-1}\chi_{\sigma'}{}^\dagger(-\mathbf{x},x^0)\,,\quad(9.43)$$

and of course it is also true of **CPT**.

X. CHIRALITY AND RENORMALIZED MASS

We have not made any distinction, either here or in Ref. 1, between the mass characterizing the free field and the mass of the physical particles. This was purposeful, because it is always possible and preferable to arrange that the unperturbed and the full Hamiltonians have the same spectrum. But there still remains the question: Under what circumstances will the physical particle mass in fact be zero? The classic conditions are gauge invariance or chirality [i.e., "γ_5"] conservation. Gauge invariance is without content for the $(j,0)$ and $(0,j)$ fields discussed in this article, so we are led to consider the implications of chirality conservation. Our work in this section is entirely academic except for $j=\frac{1}{2}$, but even in this familiar case our conclusions are not quite in accord with public opinion.

For definiteness we will understand chirality conservation as invariance under a continuous transformation

$$\varphi_\sigma(x)\to e^{i\epsilon}\varphi_\sigma(x)\,;\quad \chi_\sigma(x)\to e^{-i\epsilon}\chi_\sigma(x)\,.\quad(10.1)$$

In the $2(2j+1)$-component formalism[10] we unite the $(j,0)$ and $(0,j)$ fields $\varphi_\sigma(x)$ and $\chi_\sigma(x)$ into a $(j,0)\oplus(0,j)$ field $\psi(x)$:

$$\psi(x)=\begin{bmatrix}\varphi(x)\\ \chi(x)\end{bmatrix}\quad(10.2)$$

and we write the transformation (10.1) as

$$\psi(x)\to\exp(i\epsilon\gamma_5)\psi(x)\,.\quad(10.3)$$

$$\gamma_5\equiv\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

There are other possible discrete or continuous chirality transformations, but our discussion will apply equally to all of them.

The question, of whether chirality conservation implies zero physical mass, can be asked on two different levels:

(1) Suppose that H_0 is chosen so the interaction representation fields $\varphi_\sigma(x)$ and/or $\chi_\sigma(x)$ describe free particles of zero mass, and suppose that the interaction density $\mathcal{H}(x)$ is invariant under the transformation (10.1). Is the renormalized mass then zero in each order of perturbation theory?

(2) Suppose that there exists a unitary operator which induces the transformation (10.1) on the Heisenberg representation fields, and which leaves the physical vacuum invariant. Can we then prove anything about the physical mass spectrum?

Our answers to these two questions are (1) yes, and (2) not necessarily. Let us consider perturbation theory first. The bare momentum-space propagator of the φ_σ field is given by (7.23) as

$$S(q)=-i\Pi(q)/(q^2-i\epsilon)\,.\quad(10.4)$$

The exact propagator is

$$S'(q)=S(q)+S(q)\Sigma^{(*)}(q)S'(q)=[S^{-1}(q)-\Sigma^{(*)}(q)]^{-1}.\quad(10.5)$$

The $(2j+1)\times(2j+1)$ matrix $\Sigma^{(*)}(q)$ is the sum of all proper diagrams with one φ_σ line coming in and one going out, with no propagators on these lines. Stripping away its external propagators changes the Lorentz transformation behavior of $\Sigma_{\sigma\sigma'}{}^{(*)}$ from that of $\varphi_\sigma\varphi_{\sigma'}{}^*$ to that of $\chi_\sigma\chi_{\sigma'}{}^*$, so Lorentz invariance dictates its form as

$$\Sigma_{\sigma\sigma'}{}^{(*)}(q)=i\bar{\Pi}_{\sigma\sigma'}(q)F(-q^2)\,.\quad(10.6)$$

Using (6.8) now gives the exact propagator (10.5) as

$$S'(q)=\frac{-i\Pi(q)}{[1-(-q^2)^jF(-q^2)][q^2-i\epsilon]}\,.\quad(10.7)$$

We have not used chirality yet. In general the self-energy part $\Sigma^{(*)}(q)$, and hence the function $F(-q^2)$, may have a pole at $q^2=0$, due to graphs with one intermediate χ_σ line. But under any form of chirality conservation such graphs are forbidden. (For example, there is no neutrino χ_σ field.) Hence $F(-q^2)$ has no pole at $q^2=0$, and therefore $S'(q)$ does have such a pole, corresponding to a particle of zero renormalized mass.

[10] See Ref. 1. Many features of this formalism have been worked out independently in unpublished work by D. N. Williams.

Of course there may also be another particle with nonzero mass m given by

$$1=m^{2j}F(m^2).$$

But such a particle would have to be unstable so m would lie off the physical sheet.

Now let us turn to the second question. We assume that there exists a unitary chirality operator $X(\epsilon)$ which transforms the Heisenberg representation fields into

$$X(\epsilon)\varphi_\sigma{}^H(x)X^{-1}(\epsilon)=e^{i\epsilon}\varphi_\sigma{}^H(x)\,, \qquad (10.8)$$

$$X(\epsilon)\chi_\sigma{}^H(x)X^{-1}(\epsilon)=e^{-i\epsilon}\chi_\sigma{}^H(x)\,, \qquad (10.9)$$

and which leaves the physical vacuum invariant. It is certain that this assumption alone is not sufficient, in itself, to allow us to prove anything about physical particle masses, because we have not yet said anything to connect the fields $\varphi_\sigma(x)$ and $\chi_\sigma(x)$ with each other. For instance, we might choose $\varphi_\sigma(x)$ as $(1+\gamma_5)/2$ times the electron field, and $\chi_\sigma(x)$ as $(1-\gamma_5)/2$ times the muon field. Then (10.8) and (10.9) are obviously satisfied if we choose the chirality operator as

$$X(\epsilon)=\exp\{i\epsilon\,[\text{electron number} - \text{muon number}]\}\,. \qquad (10.10)$$

But we can hardly conclude from this that the electron or muon is massless.

Clearly, the only information that can be gleaned solely from the existence of $X(\epsilon)$ is just what would follow from any ordinary additive conservation law. Namely, the propagator of $\varphi_\sigma(x)$ or $\chi_\sigma(x)$ can receive no contribution from any *massive purely neutral* one-particle state that has no degeneracy beyond the $(2j+1)$-fold degeneracy associated with its spin.[11] For any such state $|\mathbf{p},\mu\rangle$ would have to be a chirality eigenstate

$$X(\epsilon)|\mathbf{p},\mu\rangle=e^{i\xi\epsilon}|\mathbf{p},\mu\rangle \quad (\mu=-j,\cdots,j)\,, \qquad (10.11)$$

and thus

$$\langle 0|\varphi_\sigma{}^H(x)|\mathbf{p},\mu\rangle=0 \quad \text{unless} \quad \xi=1\,, \qquad (10.12)$$

$$\langle 0|\varphi_\sigma{}^{H\dagger}(x)|\mathbf{p},\mu\rangle=0 \quad \text{unless} \quad \xi=-1\,. \qquad (10.13)$$

But **CP** or **CPT** conservation tells us that these two matrix elements are proportional to each other, and hence must both vanish. [Observe that we cannot forbid a massless purely neutral particle from contributing to the propagator of $\varphi_\sigma(x)$ or $\chi_\sigma(x)$, since **CP** and **CPT** reverse its helicity, and its two helicity states might have opposite chirality. This is consistent with the remark[6] that it is only a matter of convention whether we call a massless particle purely neutral or not.]

[11] This is an abbreviated version of a proof given by B. Touschek, in *Lectures on Field Theory and the Many-Body Problem*, edited by E. R. Caianiello (Academic Press Inc., New York, 1961), p. 173. It is not clear from Touschek's article whether he feels that this theorem implies that the neutrino cannot have finite mass. As indicated herein, I do not.

Unfortunately this theorem offers no proof that the accepted chirality-conserving weak interactions do not give a massive neutrino, with a distinct massive antineutrino. It should be kept in mind that we cannot decide just by looking at a Lagrangian whether the physical one-particle states will be purely neutral or not. Of course, any massless particle can be called purely neutral, but this is not relevant if what we want is to prove the absence of massive particles.

We can say somewhat more about the mass spectrum if we are willing to assume parity conservation [which links $\varphi_\sigma(x)$ with $\chi_\sigma(x)$ by (9.30) and (9.31)] as well as chirality conservation. In this case the propagator of $\varphi_\sigma(x)$ or $\chi_\sigma(x)$ can receive no contribution from any massive one-particle state that has no degeneracy, beyond the $(2j+1)$-fold degeneracy associated with its spin, and an additional 2-fold degeneracy if it happens to have a distinct antiparticle. For it would then be possible to form a one-particle chirality eigenstate $|\mathbf{p},\mu\rangle$:

$$X(\epsilon)|\mathbf{p},\mu\rangle=\exp(i\epsilon\xi)|\mathbf{p},\mu\rangle \qquad (10.14)$$

by taking $|\mathbf{p},\mu\rangle$ as either the one-particle state itself or some linear combination of it and its charge conjugate. Lorentz invariance requires that

$$\langle 0|\varphi_\sigma{}^H(x)|\mathbf{p},\mu\rangle=N_\varphi(2\omega)^{-1/2}D_{\sigma\mu}{}^{(j)}[L(\mathbf{p})]e^{ip\cdot x}\,, \qquad (10.15)$$

$$\langle 0|\chi_\sigma{}^H(x)|\mathbf{p},\mu\rangle=N_\chi(2\omega)^{-1/2}\bar{D}_{\sigma\mu}{}^{(j)}[L(\mathbf{p})]e^{ip\cdot x}\,. \qquad (10.16)$$

Parity conservation tells us further that

$$|N_\varphi|=|N_\chi|\equiv N\,. \qquad (10.17)$$

This is just to say that the matrix element of the $2(2j+1)$-component field $\psi(x)$ satisfies the generalized Dirac equation [Eq. (7.19) of Ref. 1], which is to be expected under the assumption of parity conservation. But (10.8) and (10.14) give $N=0$ unless $\xi=+1$, while (10.9) and (10.14) give $N=0$ unless $\xi=-1$, so we may conclude that $N=0$. Again, this proof does not apply for zero mass, because the two helicity states are unconnected by space rotations and hence may have different ξ's.

[It might at first sight appear that the free fields constructed in Ref. 1 provide a counter-example to this proof. In the absence of interactions they certainly describe nondegenerate particles with nonvanishing bare and physical masses, and yet there is no coupling that violates either parity or chirality. The trouble with this argument is that no operator $X(\epsilon)$ can be constructed; in fact Eqs. (7.23) and (7.25) of Ref. 1 show that

$$\langle T\{\varphi_\sigma(x),\chi_{\sigma'}{}^\dagger(y)\}\rangle_0\neq 0\,. \qquad (10.18)$$

This point is more transparent in the conventional language in which we would just say that the free-field Lagrangian does not conserve chirality. As $m\to 0$, (10.18) vanishes as m^{2j}, and for $m=0$ it is easy to construct $X(\epsilon)$ explicitly.]

The last proof is of some interest, because it shows that unless the vacuum or electron is degenerate, the mass of the electron cannot arise entirely from electromagnetic interactions, which conserve both parity and chirality. But it is useless for the neutrino, and we are forced to conclude that only perturbation theory can account for its zero mass.

XI. CONCLUSIONS

The Feynman rules for massless particles in the $(2j+1)$-component formalism are identical with those derived in Ref. 1 for particles of mass $m>0$. It is only necessary to pass to the limit $m\rightarrow 0$ to obtain the correct propagators for internal lines, and wave functions for external lines. Also, the various possible invariant Hamiltonians $\mathcal{H}(x)$ can be constructed out of the fields $\varphi_\sigma(x)$ and $\chi_\sigma(x)$, with no distinction between massive and massless particle fields.

Furthermore, the transformation properties of $\varphi_\sigma(x)$ and $\chi_\sigma(x)$ under **T**, **C**, and **P** are the same for $m>0$ and $m=0$. If **P** and/or **C** are conserved it is very convenient to unite $\varphi_\sigma(x)$ and $\chi_\sigma(x)$ into a $2(2j+1)$-component field $\psi(x)$, which transforms according to the reducible $(j,0)\oplus(0,j)$ representation; for $j=\frac{1}{2}$ this yields the Dirac formalism, while for $j=1$ it corresponds to the union of the irreducible fields $\mathbf{E}\pm i\mathbf{B}$ into a six-vector $\{\mathbf{E},\mathbf{B}\}$. Here again there is no distinction to be made between zero and nonzero mass, so we need not repeat here the details of the $2(2j+1)$-component formalism[10] constructed in Ref. 1.

We have seen no hint of anything like gauge invariance in our work so far. In fact, the really significant distinctions between field theories for zero and nonzero mass arise when we try to go beyond the $(2j+1)$- or $2(2j+1)$-component formalisms. In particular, for $m>0$ there is no difficulty in constructing tensor fields transforming according to the $(j/2,j/2)$ representations, while for $m=0$ this is strictly forbidden by the theorem proven in Sec. III. We will see in a forthcoming article that the attempt to evade this prohibition and yet keep the S matrix Lorentz-invariant yields all the results usually associated with gauge invariance.

PHYSICAL REVIEW VOLUME 134, NUMBER 4B 25 MAY 1964

Possible Effects of Strong Interactions in Feinberg-Pais Theory of Weak Interactions. II

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(Received 23 December 1963)

In a previous paper, a simplified model was used to study the effects of strong interactions on the weak interaction theory of Feinberg and Pais. In this paper, we use a more general argument, a power count based upon the Ward-Takahashi-Nishijima multimeson vertex function identity, to show that the same conclusion remains valid even when crossed ladder graphs are included. Our conclusion may not apply, however, to the modified program of peratization where $W-W$ scattering plays an essential role.

I. INTRODUCTION

In a previous paper,[1] the possible effects of strong interactions on the peratization theory of Feinberg and Pais[2] were studied in a simplified model where the strong interactions acted through modifications only of the baryon vertices and propagators. It was shown there that the final "peratized" nuclear vector β-decay coupling strength $G_\beta{}^v$ is no longer equal to the "peratized" μ-decay coupling strength, G_μ if the vector current is conserved. In this paper, we wish to present an argument which shows that the same power counting conclusion holds when all possible effects of strong interactions, within the framework of peratization theory, are taken into account. Furthermore, the very nature of our argument shows that the same conclusion holds even when one includes, in peratization theory, the sum over the crossed ladder graphs so long as power counting is valid. That is to say, if we define the peratized (crossed+uncrossed ladder graphs) μ-decay constant by $G_\mu=(g^2/m^2)(1-\eta)$, then the corresponding peratized nuclear vector β-decay constant is $G_\beta{}^v=(g^2/m^2)(1-Z\eta)$, where Z is the strong interaction nucleon renormalization factor. Thus, unless peratization vanishes ($\eta=0$) when all graphs are included, the situation remains that $G_\mu\neq G_\beta{}^v$ when the vector current is conserved. This makes it hard to understand the

* Supported in part by a grant from the National Science Foundation.
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